# No quartic curve supports 3-body figure-eight solution under homogeneous potential

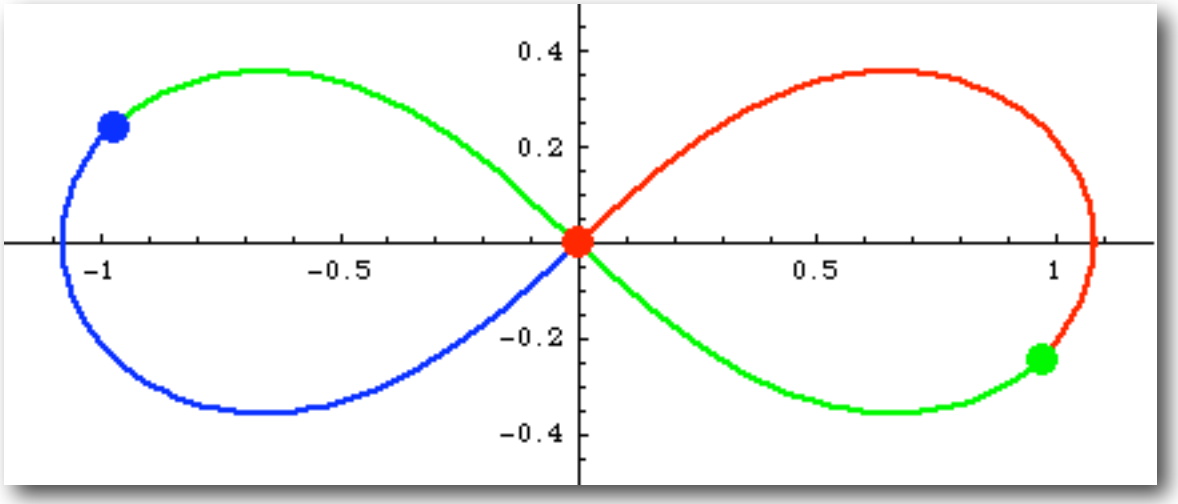
Toshiaki Fujiwara

Hamsys-2008, July 9

CIMAT, Guanajuato

# Three body figure-eight solution

- C. Moore (1993) found numerically
- A. Chenciner and R. Montgomery (2000) proved the existence



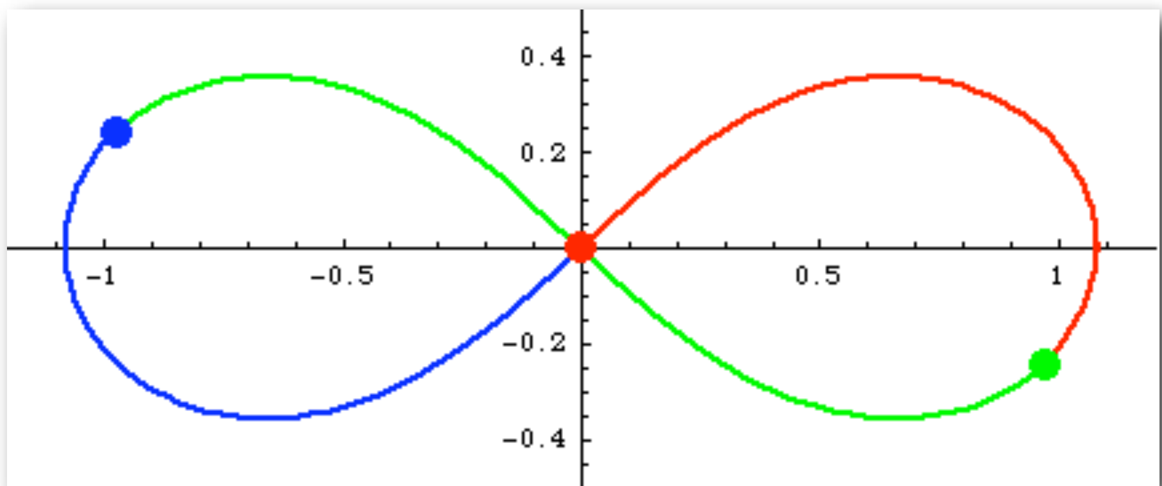
# Three body figure-eight solution

$$i = 1, 2, 3, m_i = 1$$

$$\ddot{q}_k = \sum_{j \neq k} \frac{q_j - q_k}{|q_j - q_k|^3},$$

$$\begin{cases} q_1(t) = q(t), \\ q_2(t) = q(t + T/3), \\ q_3(t) = q(t + 2T/3), \end{cases}$$

$$\sum_i q_i = 0, \sum_i q_i \times \dot{q}_i = 0$$

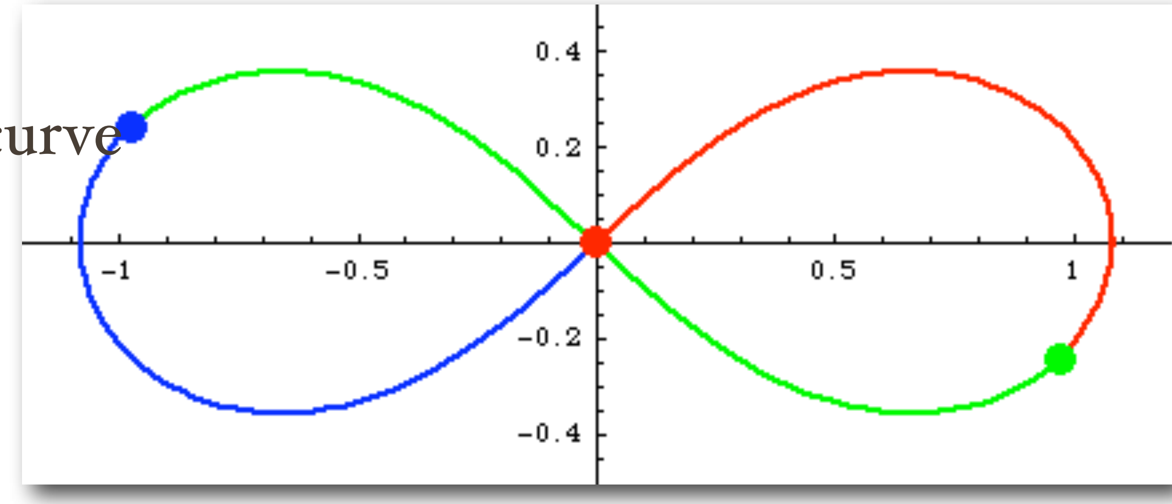


# Three body figure-eight solution

- C. Simó (2001) : Tried to fit the orbit by order 4, 6, 8 curve, numerically.

$$f(x,y) = \sum_{k=1}^{m} \sum_{j=0}^{k} c_{2(k-j)} x^{2(k-j)} y^{2j} = 0$$

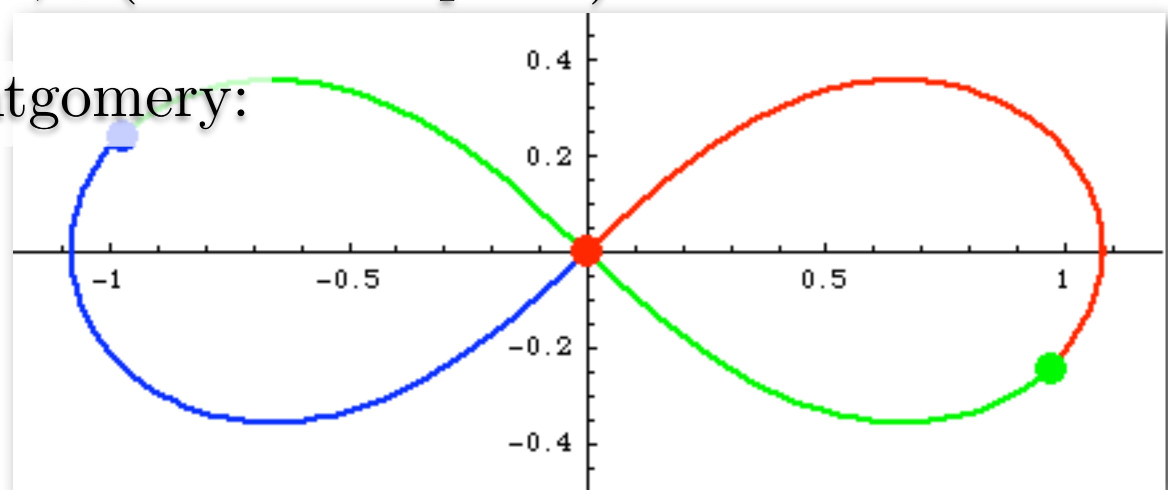
He conclude that the curve is not order 4,6,8.




"Even with $m = 2$ (order 4) one cannot distinguish the eight from $f(x, y) = 0$ without magnification."

# A few is known about the shape of the figure-eight solution

- A. Chenciner and R. Montgomery:
  The shape is star. Namely $r = r(\theta)$.
- C. Simó:
  The curve is not order 4, 6, 8.(Numerical proof)
- T. Fujiwara and R. Montgomery:
  Each lobe is convex.

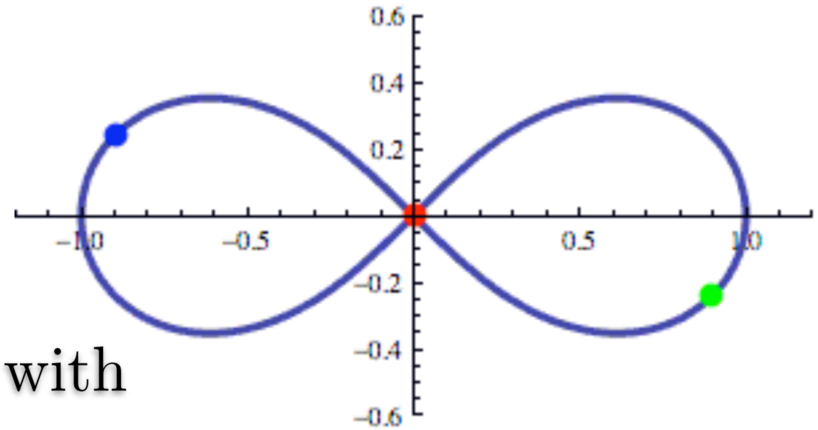



I want to know more about the shape.

# Three body figure-eight solution on the Lemniscate

- T. Fujiwara, H. Fukuda, H. Ozaki (2003)

$$(x^2+y^2)^2 = x^2 - y^2$$




The motion satisfies $\ddot{q}_i = -\frac{\partial}{\partial q_i} U$ with

$$U = \sum_{i<j} \left( \frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2 \right),$$

an inhomogeneous potential.

# The shape of figure-eight orbit

| | |
|---|---|
| is NOT order 4, 6, 8 | for $U = -\sum \frac{1}{r_{ij}}$ |
| IS order 4 (Lemniscate) | for $U = \sum \frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2$ |

My question is;
Is there any figure-eight solution under the homogeneous potential $U = \frac{1}{\alpha} \sum r_{ij}^{\alpha}$, $\alpha \neq 0$ or $U = \sum \log r_{ij}$, whose orbit is order 4 curve?

$H = K + U$, $\ddot{q}_i = -\frac{\partial U}{\partial q_i}$. $\alpha = -1$ for the Newton potential.

# The shape of figure-eight orbit

| | |
|---|---|
| is NOT order 4, 6, 8 | for $U = -\sum \frac{1}{r_{ij}}$ |
| IS order 4 (Lemniscate) | for $U = \sum \frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2$ |

In other words;
Can quartic curve (order 4 curve) support the figure-eight solution under the potential $\frac{1}{\alpha} \sum r_{ij}^{\alpha}$ or $\sum \log r_{ij}$?

# The shape of figure-eight orbit

| | |
|---|---|
| is NOT order 4, 6, 8 | for $U = -\sum \frac{1}{r_{ij}}$ |
| IS order 4 (Lemniscate) | for $U = \sum \frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2$ |

The answer is;
Quartic curve (order 4 curve) cannot support the figure-eight solution under the potential $\frac{1}{\alpha} \sum r_{ij}^{\alpha}$ or $\sum \log r_{ij}$.

| | |
|---|---|
| is not order 4 | for $U = \frac{1}{\alpha} \sum r_{ij}^{\alpha}$ or $\sum \log r_{ij}$ |

# We will show this in two steps

is not order 4 for $U = \frac{1}{\alpha}\sum r_{ij}^{\alpha}$ or $\sum \log r_{ij}$

Quartic

↓

"Lemniscate"

↓

✕

for

homogeneous U

1. The order 4 curve that can support a figure-eight solution is the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ and $x \to \lambda x$, $y \to \mu y$.

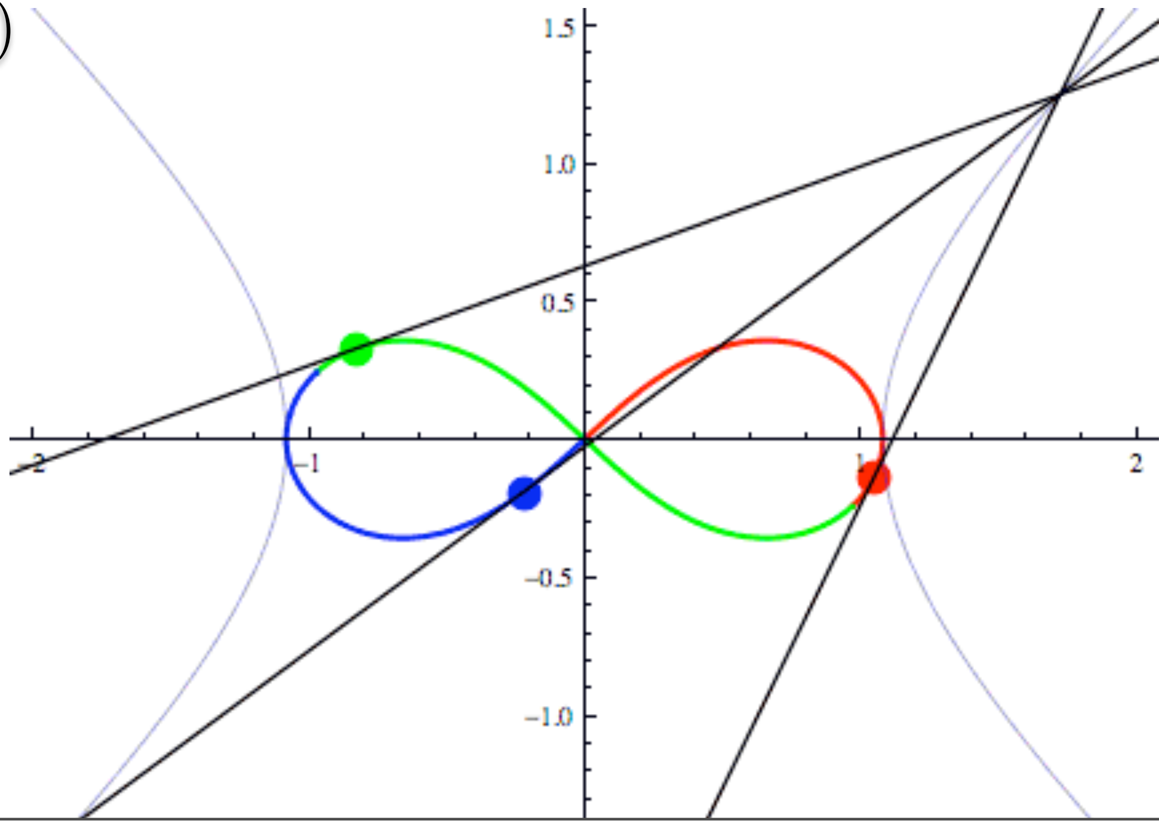
2. The lemniscate $(x^2+y^2)^2 = x^2 - y^2$ and $x \to \lambda x, y \to \mu y$ cannot support the figure-eight solution under the potential $\frac{1}{\alpha}\sum r_{ij}^{\alpha}$, $\alpha \neq 0$ or $\log \sum r_{ij}$.

# The first step

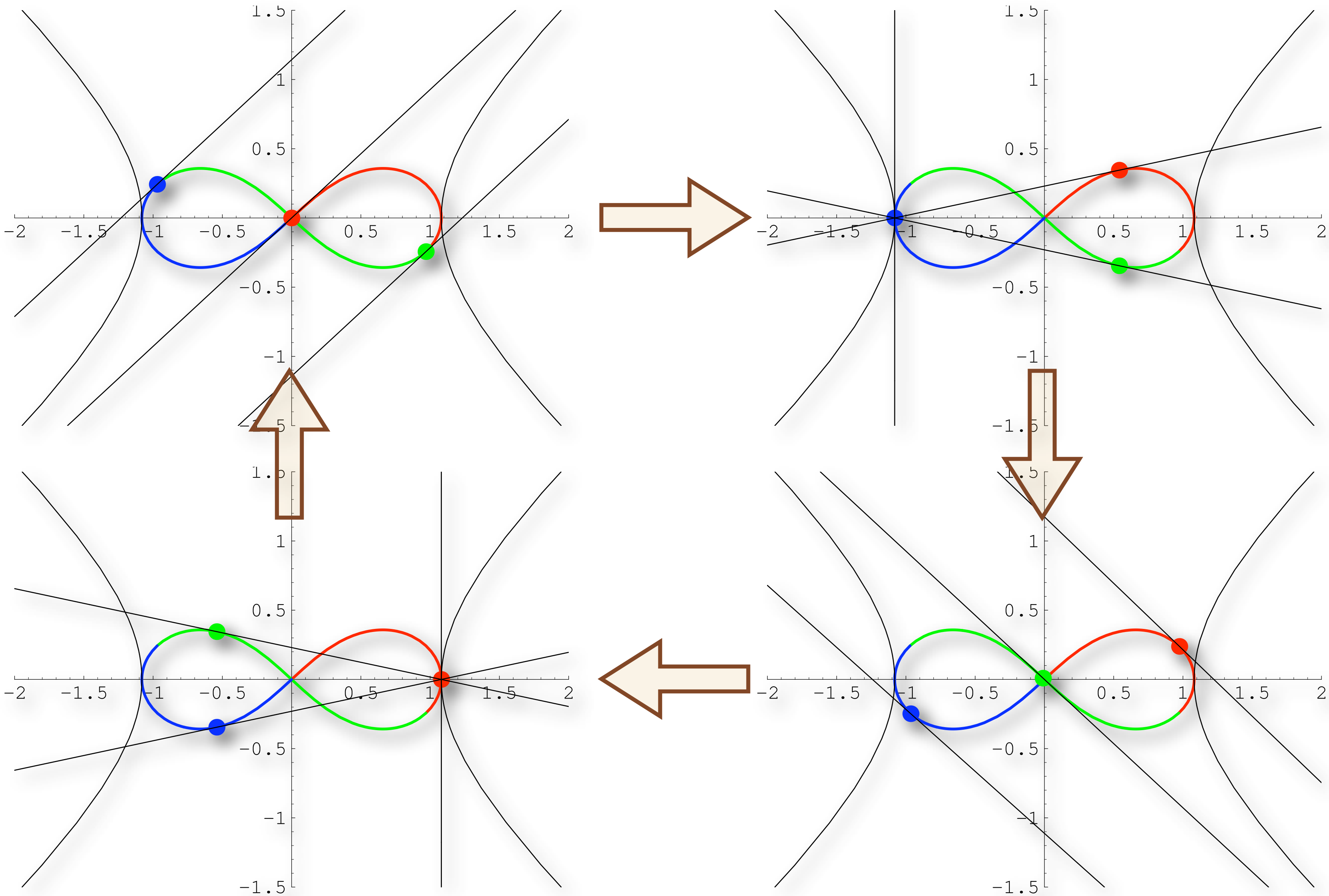
Quartic curve that can support figure-eight solution is only the lemniscate and its scale variant

# Three tangents theorem

For three-body motion, if $\sum_{i=1,2,3} p_i = 0$ and $\sum_{i=1,2,3} q_i \times p_i = 0$, then three tangent lines meet at a point for each instant. (Fujiwara, Fukuda and Ozaki 2003)

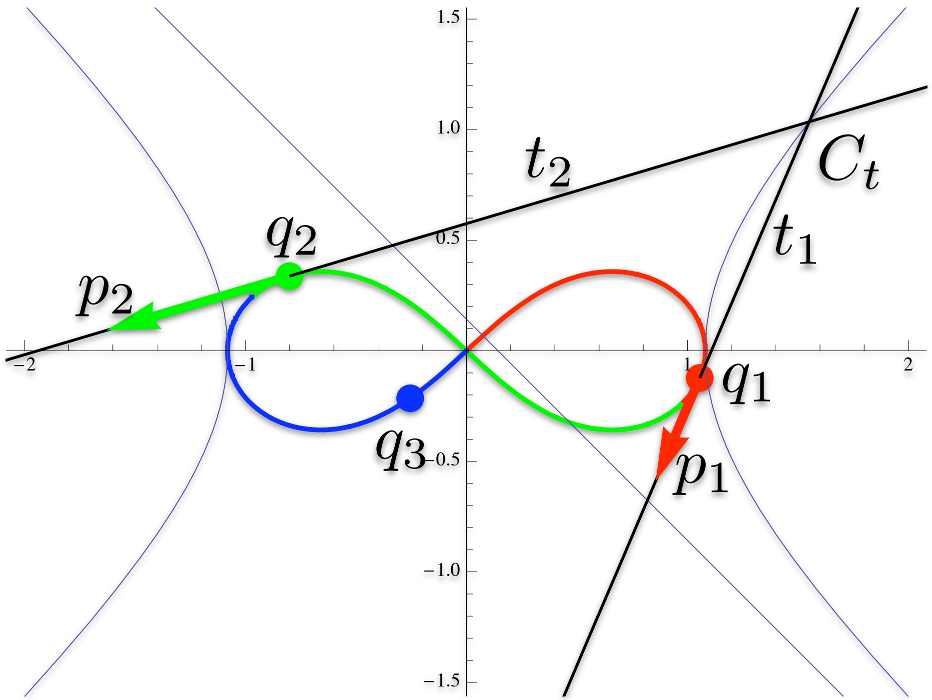


# Euler & Isosceles Config.



# Proof

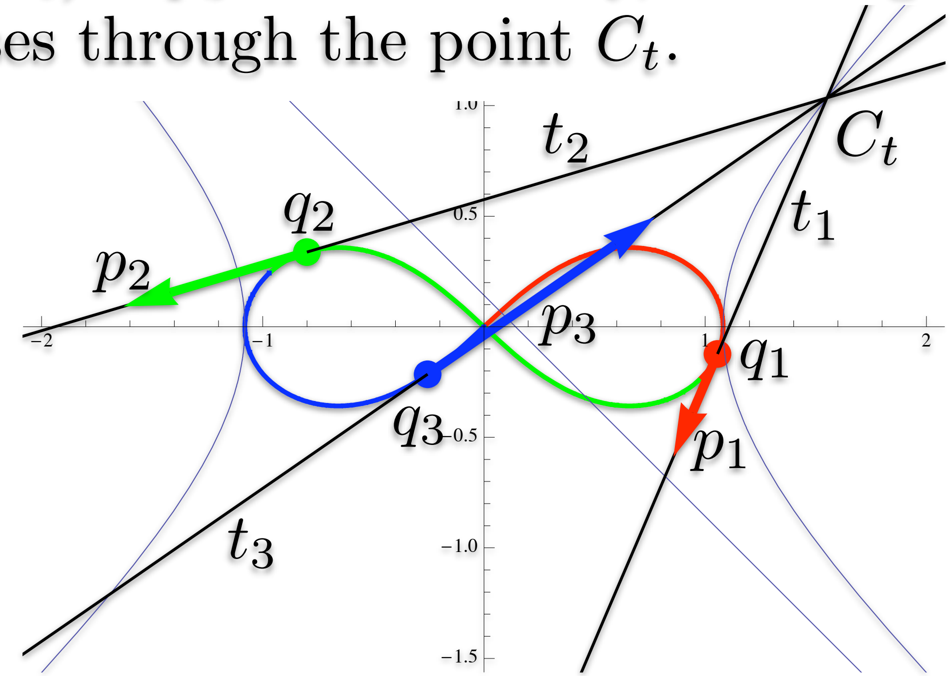
If two tangents $t_1$ and $t_2$ meet at a point $C_t$, then $\sum p_i = 0$ and $\sum q_i \times p_i = 0 \Rightarrow \sum (q_i - C_t) \times p_i = 0$ and $(q_1 - C_t) \times p_1 = 0$, $(q_2 - C_t) \times p_2 = 0$.



# Proof

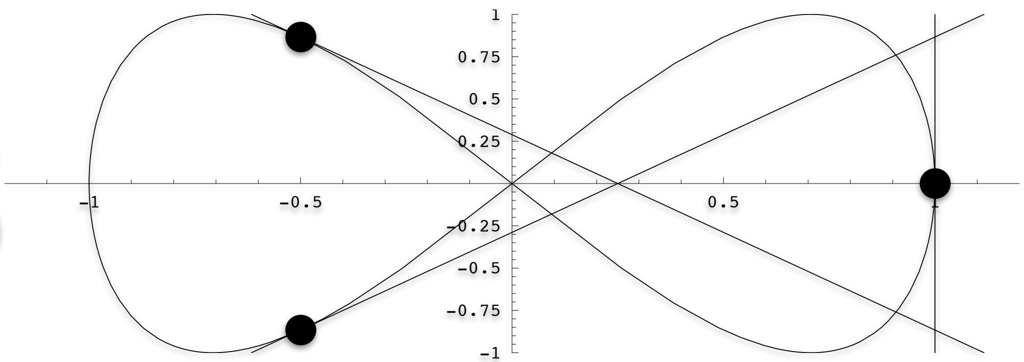
If two tangents $t_1$ and $t_2$ meet at a point $C_t$, then $\sum p_i = 0$ and $\sum q_i \times p_i = 0 \Rightarrow \sum (q_i - C_t) \times p_i = 0$ and $(q_1 - C_t) \times p_1 = 0$, $(q_2 - C_t) \times p_2 = 0$.

$\therefore (q_3 - C_t) \times p_3 = 0$. Namely, the tangent line $t_3$ also passes through the point $C_t$.



# 3 tangent theorem gives a criterion for the shape

$$\begin{cases} x = \cos(\theta), \\ y = \sin(2\theta) \end{cases}$$

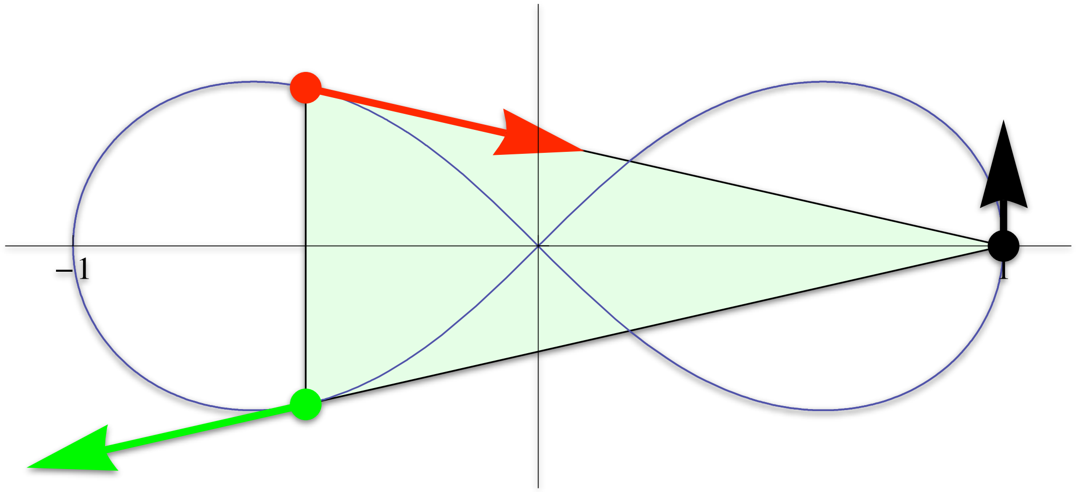



Three tangent lines for this curve at the isosceles configuration do not meet at a point.

Therefore, this curve cannot support

$$\sum_{i=1,2,3} q_i \times p_i = 0 \text{ motion.}$$

# Isosceles configuration

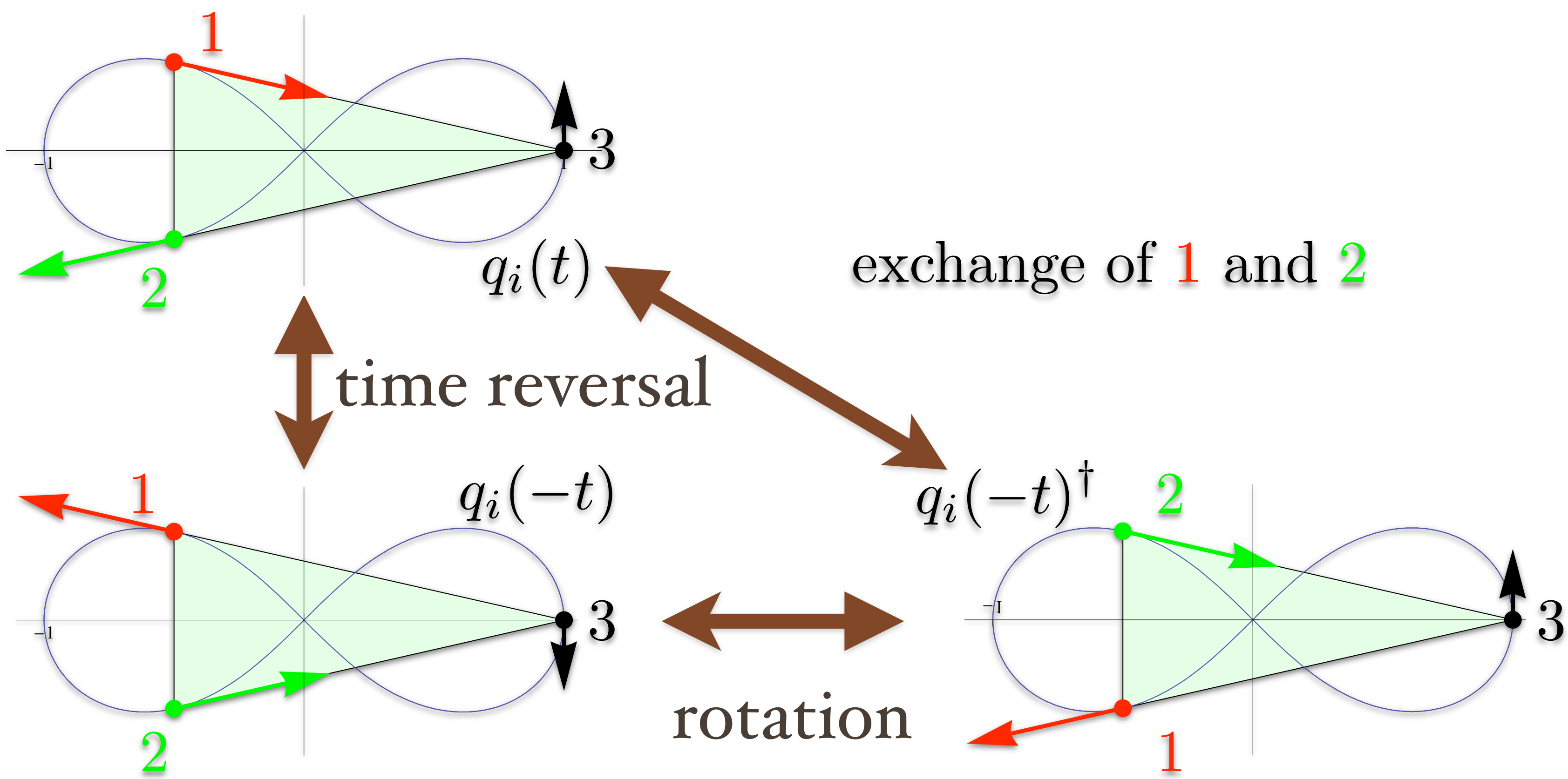



$$q_1(0) = (-1/2, y_0), q_2(0) = (-1/2, -y_0), q_3(0) = (1, 0).$$

$$p_1(0) = \left(\frac{3v}{2y_0}, -v\right), p_2(0) = \left(-\frac{3v}{2y_0}, -v\right), p_3(0) = (0, 2v)$$

Two parameters $y_0$ and $v$.

# We assume the invariance under time reversal, rotation and exchange of bodies




$\therefore\ q_2(t) = q_1(-t)^\dagger$ and $q_3(t) = q_3(-t)^\dagger$, where $(x, y)^\dagger = (x, -y)$.

$q_3$ is determined by $q_3(t) = -\Big(q_1(t) + q_2(t)\Big) = -\Big(q_1(t) + q_1(-t)^\dagger\Big)$.

# Power series of the orbit

$$
\begin{aligned}
q_1(t) &= \Big(-\frac{1}{2}, y_0\Big) + \Big(\frac{3v}{2y_0}, -v\Big)t + \Big(\alpha_x, \alpha_y\Big)\frac{t^2}{2} + \Big(\beta_x, \beta_y\Big)\frac{t^3}{6} + O(t^4) \\
&\Downarrow \\
&\Downarrow \quad q_2(t) = q_1(-t)^\dagger,\ q_3(t) = -(q_1(t) + q_1(-t)^\dagger) \\
&\Downarrow \\
q_2(t) &= \Big(-\frac{1}{2}, -y_0\Big) + \Big(-\frac{3v}{2y_0}, -v\Big)t + \Big(\alpha_x, -\alpha_y\Big)\frac{t^2}{2} + \Big(-\beta_x, \beta_y\Big)\frac{t^3}{6} + O(t^4) \\
q_3(t) &= \Big(1, 0\Big) + \Big(0, 2v\Big)t + \Big(-2\alpha_x, 0\Big)\frac{t^2}{2} + \Big(0, -2\beta_y\Big)\frac{t^3}{6} + O(t^4)
\end{aligned}
$$

We actually need this form and vanishing angular momentum.

$$c = \sum_i q_i \times p_i = \left(3v\alpha_x - y_0\beta_x + \frac{3v\alpha_y}{2y_0} - \frac{3\beta_y}{2}\right)t^2 + O(t^3)$$

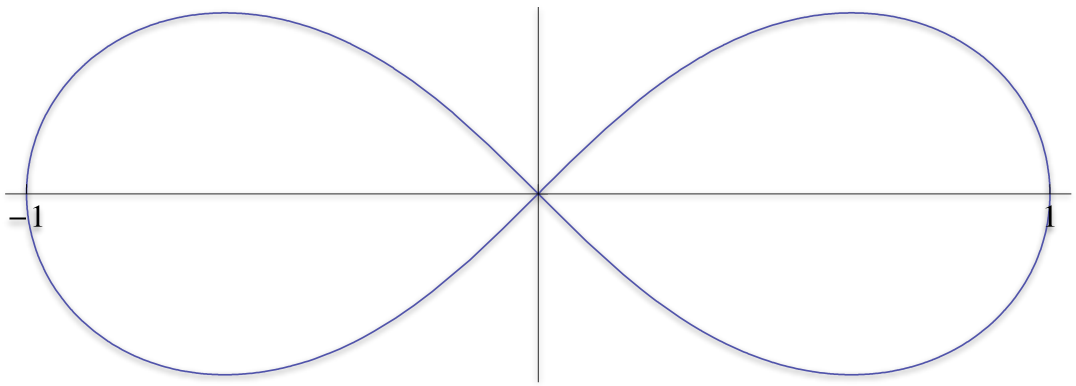
$$\therefore c = \sum q_i \times p_i = 0$$

$$\Downarrow$$

$$3v\alpha_x - y_0\beta_x + \frac{3v\alpha_y}{2y_0} - \frac{3\beta_y}{2} = 0.$$

# Quartic curves

$$P(x^2, y^2) = x^4 + ax^2y^2 + by^4 - x^2 + y^2 = 0$$




$$P\Big(x_i(t)^2, y_i(t)^2\Big) = Q_i(t) = 0, \text{ then}$$

$$Q_i(0) = \frac{dQ_i}{dt}(0) = \frac{d^2Q_i}{dt^2}(0) = \cdots = \frac{d^nQ_i}{dt^n}(0) = 0$$

$$Q_i(t) = P\Big(x_i(t)^2, y_i(t)^2\Big) \Rightarrow Q_1(t) = Q_2(t)$$

$$Q_1(0) = 0 \Rightarrow \frac{1}{16}\Big(-3 + 16y_0^2 + 4ay_0^2 + 16by_0^4\Big) = 0.$$

$$\frac{dQ_1}{dt}(0) = 0 \Rightarrow -\frac{v}{4y_0}\Big(-3 + 8y_0^2 + 8ay_0^2 + 16by_0^4\Big) = 0.$$

$$\Downarrow$$

$$\therefore a = 2$$

$$b = \frac{3(1 - 8y_0^2)}{16y_0^4}$$

$$P = x^4 + 2x^2y^2 + by^4 - x^2 + y^2 = 0$$

$$\frac{d^2Q_3}{dt^2} = 0, \;\; Q_3 = P\Big(x_3(t)^2, y_3(t)^2\Big)$$

$$\Downarrow$$

$$2(6v^2 - \alpha_x) = 0$$

$$\therefore \alpha_x = 6v^2$$

$$\frac{d^2Q_1}{dt^2}(0) = 0 \Rightarrow 2v^2(3 + 6y_0^2 - 8y_0^4) + y_0\alpha_y(1 - 4y_0^2) = 0,$$

$$\frac{d^3Q_1}{dt^3}(0) = 0 \Rightarrow 3v^2(5 - 16y_0^4) + 2y_0\alpha_y(1 - 4y_0^2) = 0$$

$$\Downarrow$$

$$\therefore 3(1 - 8y_0^2) = 16y_0^4$$

Since $b = \dfrac{3(1-8y_0^2)}{16y_0^4}$, we get $b = 1$.

Thus, $P(x^2, y^2) = x^4 + 2x^2y^2 + 1y^4 - x^2 + y^2 = 0.$

Therefore, quartic orbit that can support figure-eight solution is only the lemniscate,

$$(x^2 + y^2)^2 = x^2 - y^2$$

and $x \to \lambda x$, $y \to \mu y$.

Actually, FFO found a figure-eight solution on the lemniscate. The motion is governed by a inhomogeneous potential.

# The second step

"Lemniscate" cannot support the figure-eight solution under homogeneous potential

# Motion under homogeneous potential

Now, we assume the equation of motion,

$$\frac{d^2 q_i(t)}{dt^2} = \sum_j (q_j - q_i)|q_j - q_i|^{2\gamma}, \ \gamma \in \mathbb{R}$$

$\gamma = -3/2$ for the Newton force

Then, we will show that the orbit $q_i = (x_i, y_i)$ cannot satisfy $(x^2 + \mu^2 y^2)^2 = x^2 - \mu^2 y^2$,
even if we tune $\gamma \in \mathbb{R}$ and the initial parameters $y_0 \in \mathbb{R}$ and $v \in \mathbb{R}$

# Power series of the orbit

$$x_1(t) = -\frac{1}{2} + \frac{3vt}{2y_0} + \frac{3}{4}\left(y_0^2 + \frac{9}{4}\right)^{\gamma} t^2 - \frac{v}{8y_0}\left(2^{2\gamma+2}\left(y_0^2\right)^{\gamma} + \frac{2\left(y_0^2+\frac{9}{4}\right)^{\gamma}\left(24\gamma y_0^2 + 4y_0^2 + 18\gamma + 9\right)}{4y_0^2+9}\right) t^3 + \dots$$

$$y_1(t) = y_0 - vt - \left(2^{2\gamma} y_0 \left(y_0^2\right)^{\gamma} + \frac{1}{2} y_0 \left(y_0^2 + \frac{9}{4}\right)^{\gamma}\right) t^2 + \frac{v\left(y_0^2+\frac{9}{4}\right)^{\gamma}\left((8\gamma+4)y_0^2 + 6\gamma + 9\right) t^3}{2\left(4y_0^2+9\right)} + \dots$$

$$x_2(t) = -\frac{1}{2} - \frac{3vt}{2y_0} + \frac{3}{4}\left(y_0^2 + \frac{9}{4}\right)^{\gamma} t^2 + \frac{v}{8y_0}\left(2^{2\gamma+2}\left(y_0^2\right)^{\gamma} + \frac{2\left(y_0^2+\frac{9}{4}\right)^{\gamma}\left(24\gamma y_0^2 + 4y_0^2 + 18\gamma + 9\right)}{4y_0^2+9}\right) t^3 + \dots$$

$$y_2(t) = -y_0 - vt + \left(2^{2\gamma} y_0 \left(y_0^2\right)^{\gamma} + \frac{1}{2} y_0 \left(y_0^2 + \frac{9}{4}\right)^{\gamma}\right) t^2 + \frac{v\left(y_0^2+\frac{9}{4}\right)^{\gamma}\left((8\gamma+4)y_0^2 + 6\gamma + 9\right) t^3}{2\left(4y_0^2+9\right)} + \dots$$

$$x_3(t) = 1 - \frac{3}{2}\left(y_0^2 + \frac{9}{4}\right)^{\gamma} t^2 + 0 \times t^3 + \dots$$

$$y_3(t) = 2vt - \frac{v\left(y_0^2+\frac{9}{4}\right)^{\gamma}\left((8\gamma+4)y_0^2 + 6\gamma + 9\right) t^3}{4y_0^2+9} + \dots$$

Here, I write the series to $t^3$, but actually, we calculated to the order $t^6$ using Mathematica.

# Power series of the condition

$$\left(x_i(t)^2+\mu^2 y_i(t)^2\right)^2 - x_i(t)^2+\mu^2 y_i(t)^2 = \sum_{i,n} c_{in} t^n = 0$$

with $\mu^2 = (2\sqrt{3}-3)/(4y_0^2)$

$$\Updownarrow$$

$c_{in} = 0$ for $i = 1, 2, 3$ and $n = 0, 1, 2, \ldots$

Conditions for the parameters $y_0$, $v$, $\gamma$.

$$c_{i0} = c_{i1} = 0$$

$$c_{12} = \frac{3}{2y_0^2}\left(\sqrt{3}v^2 - 2^{2\gamma}(2-\sqrt{3})y_0^{2(\gamma+1)}\right) = 0,$$

$$c_{32} = \frac{3\sqrt{3}\left(2-\sqrt{3}\right)v^2}{y_0^2} - 3\left(y_0^2 + \frac{9}{4}\right)^{\gamma} = 0$$

Therefore,

$$v^2 = \frac{2-\sqrt{3}}{\sqrt{3}} 2^{2\gamma} y_0^{2(\gamma+1)},$$

$$\left(\frac{9}{4} + y_0^2\right)^{\gamma} = \left(2-\sqrt{3}\right)^2 (2y_0)^{2\gamma}.$$

$$c_{14} = 0 \Leftrightarrow \gamma = \frac{32 y_0^2 \left(4 y_0^2 + 9\right)}{16 \left(6 - \sqrt{3}\right) y_0^4 + 216 y_0^2 - 27\sqrt{3}}.$$

# Finally, we get two equations for $y_0$.

$$c_{34}=0 \Leftrightarrow f(y_0)=1024\left(-59+34\sqrt{3}\right)y_0^{10}-768\left(-1205+696\sqrt{3}\right)y_0^{8}$$
$$-384\left(-11115+6418\sqrt{3}\right)y_0^{6}-864\left(-6249+3608\sqrt{3}\right)y_0^{4}$$
$$-972\left(-1971+1138\sqrt{3}\right)y_0^{2}-2187\left(-97+56\sqrt{3}\right)$$
$$=0.$$

$$c_{16}=0 \Leftrightarrow g(y_0)=262144\left(-136946+79063\sqrt{3}\right)y_0^{18}+589824\left(-271018+156463\sqrt{3}\right)y_0^{16}$$
$$-393216\left(-20856+12083\sqrt{3}\right)y_0^{14}-1769472\left(-709799+409818\sqrt{3}\right)y_0^{12}$$
$$-497664\left(-5651186+3262795\sqrt{3}\right)y_0^{10}-5598720\left(-448730+259083\sqrt{3}\right)y_0^{8}$$
$$-20155392\left(-45337+26176\sqrt{3}\right)y_0^{6}-7558272\left(-17492+10097\sqrt{3}\right)y_0^{4}$$
$$-6377292\left(-1106+639\sqrt{3}\right)y_0^{2}-14348907\left(-26+15\sqrt{3}\right)$$
$$=0.$$

The two equations $f(y_0)=0$ and $g(y_0)=0$ must have common solution $y_0$. But ...

But the resultant $R(f(y_0), g(y_0))$ has value

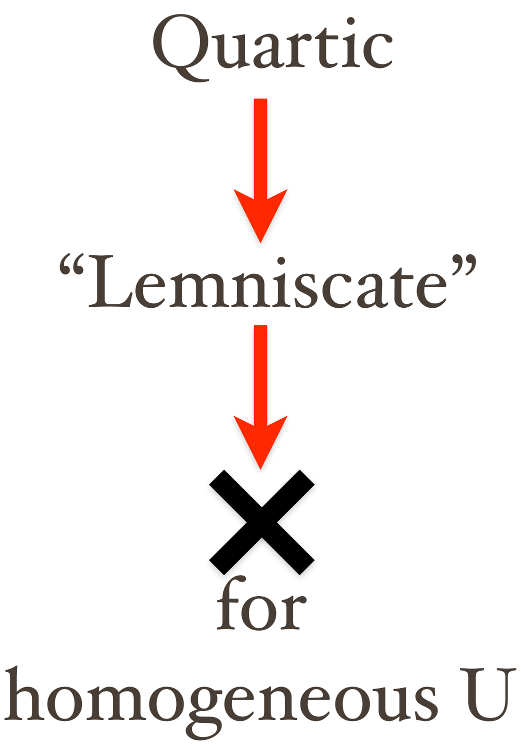
$$R\Big(f(y_0), g(y_0)\Big) = \Big(4817931$$
$$8710830100\ 6358074282$$
$$4414372772\ 4765214133$$
$$2713750858\ 6702543977$$
$$9622670249\ 7878148916$$
$$8041772964\ 5052428288$$
$$-2781634$$
$$2627070531\ 9594515840$$
$$0325073497\ 5148633067$$
$$4256800645\ 1267502319$$
$$3201035465\ 3986355344$$
$$7685710908\ 1311150080\sqrt{3}\Big)^2$$
$$= \Big(-6.0347161337844731247 \times 10^{51}\Big)^2$$
$$\neq 0.$$

Therefore, there is no solution for $f(y_0) = 0$ and $g(y_0) = 0$.

Therefore, there is no figure-eight solution whose orbit is $(x^2 + \mu^2 y^2)^2 = x^2 - \mu^2 y^2$ and satisfy the equation of motion

$$\frac{d^2 q_i(t)}{dt^2} = \sum_j (q_j - q_i)|q_j - q_i|^{2\gamma}, \text{ with } \gamma \in \mathbb{R}$$

# Conclusion




1. Quartic curve that can support figure-eight is only the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ and $x \to \lambda x$, $y \to \mu y$.

2. The lemniscate and its scale variant cannot support the figure-eight under the homogeneous potential $\frac{1}{\alpha}\sum r_{ij}^{\alpha}$ or $\sum \ln r_{ij}$.

3. Therefore, no quartic curve support figure-eight solution the homogeneous potential.

# Thank you!