Saari's conjecture for equal mass planar 3-body problem under the Newton gravity

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2012/01/07 Karuizawa

Saari's conjecture

If configurational measure $\mu = I^{\alpha/2}\,U =$ constant, then the motion is homographic. [Donald Saari 2005].

$$I = \left(\sum m_k\right)^{-1} \sum_{1 \le i < j \le N} m_i m_j |q_i - q_j|^2,$$

$$U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|q_i - q_j|^\alpha}$$

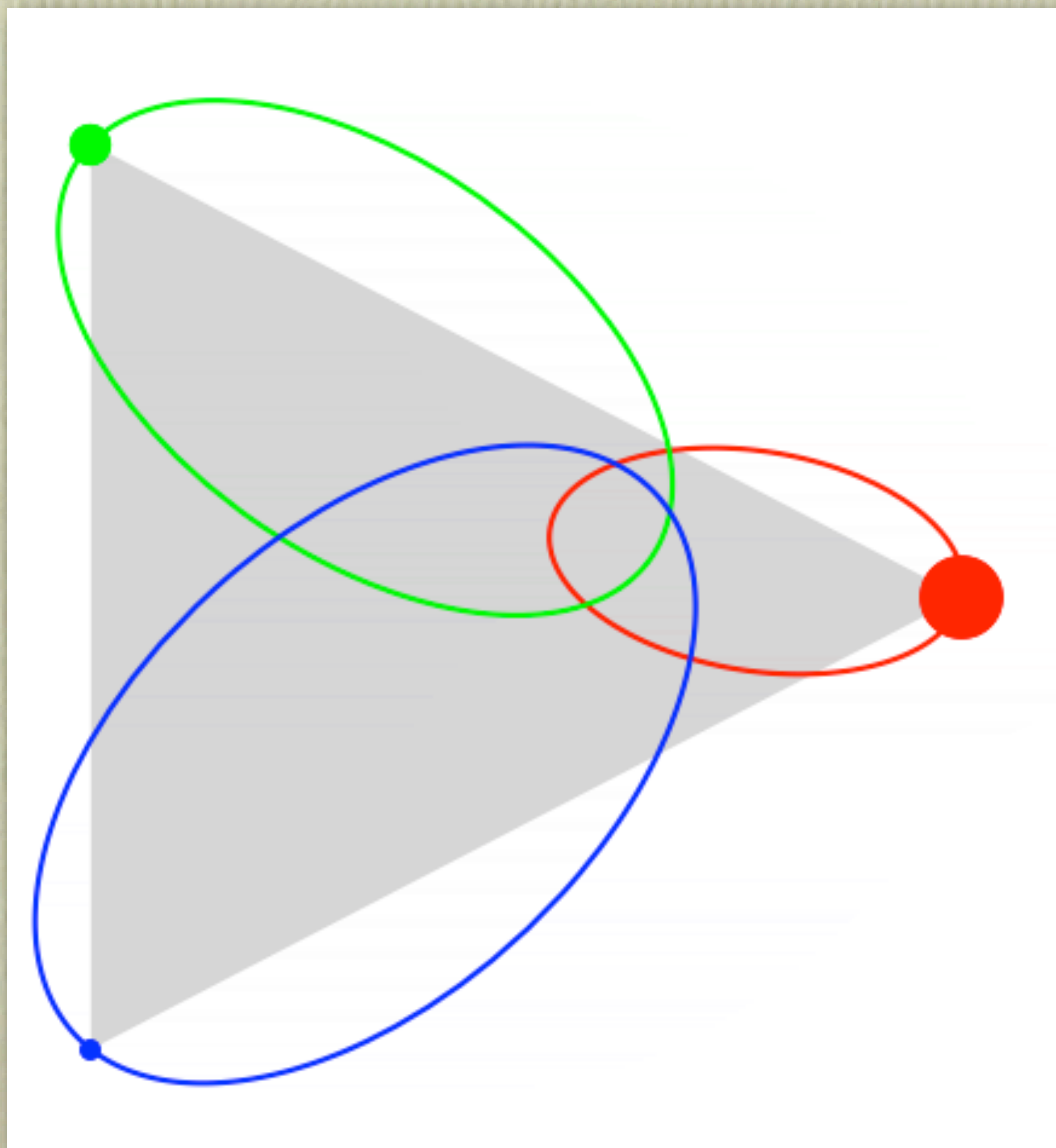
N-body $\to$ 3-body

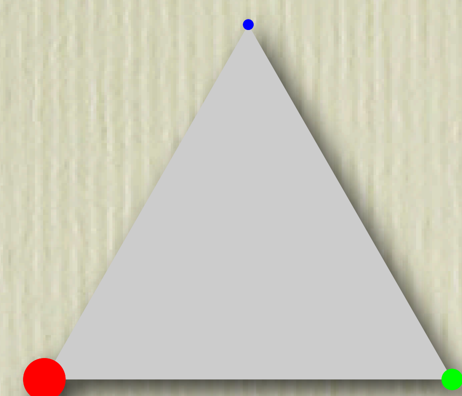
$q_k \in \mathbb{R}^3 \to \mathbb{R}^2$: planar

$m_k \to 1$: equal mass

$\alpha > 0 \to 1$: Newton

Lagrange solution

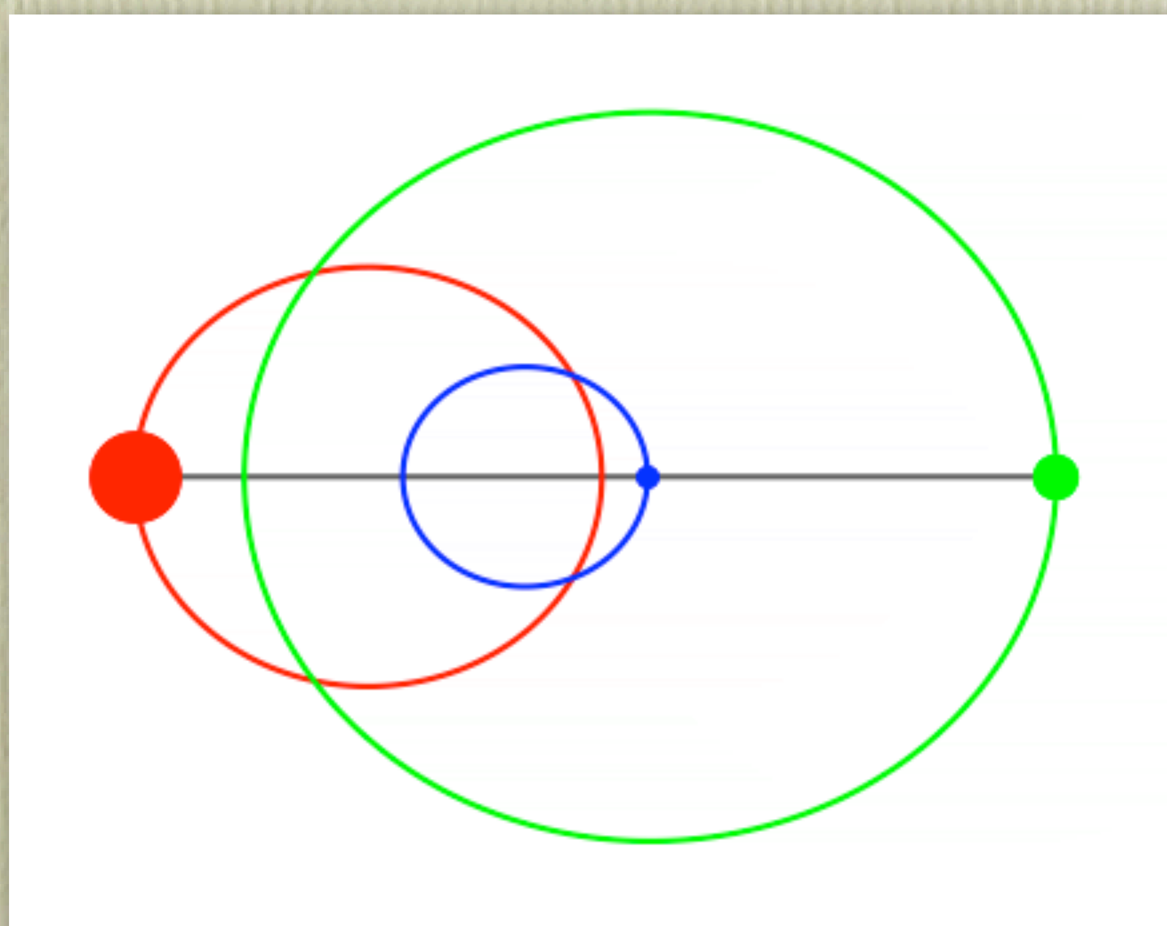




the shape is unchanged,
namely,
the motion is homographic

$$\mu = \sqrt{I}U = \text{ constant}$$

Euler solution



the shape is unchanged,
namely,
the motion is homographic

$$\mu = \sqrt{I U} = \text{ constant}$$

Saari's conjecture

$$\Downarrow$$

attention to
Shape variable

- what is the Shape variable ?
- equation of motion for the Shape variable ?

Degrees of freedom

$$q_1, q_2, q_3 \in \mathbb{C} \Rightarrow 6$$

$$\text{center of mass} \Rightarrow 2$$

$$\text{size} \Rightarrow 1$$

$$\text{rotation} \Rightarrow 1$$

$$\therefore \ \text{shape} \Rightarrow 2$$

Shape variable

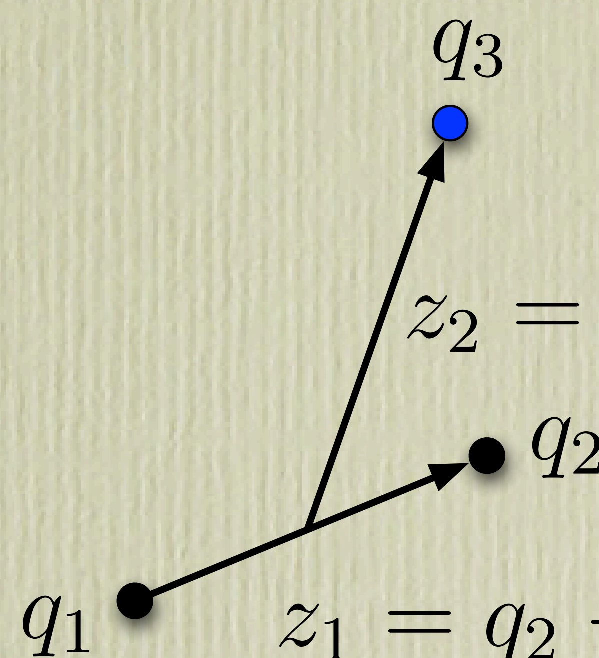
January 18, 2011, I received a mail from Richard Montgomery with an unpublished preprint [2007, unpublished].

In the preprint, Moeckel and Montgomery ...

- **the Shape variable for Planar 3-body,**
- **the Lagrangian,**
- **the equations of motion.**

The Shape variable

$$m_k = 1,$$
$$q_1 + q_2 + q_3 = 0$$



$$z_2 = q_3 - \frac{q_1 + q_2}{2} = \frac{3}{2} q_3.$$

$$z_1 = q_2 - q_1$$

Shape variable: $\zeta = \frac{z_2}{z_1} = \frac{3}{2} \frac{q_3}{q_2 - q_1}.$

Moeckel & Montgomery 2007

ζ is invariant under scaling and rotation: $q_k \to \lambda e^{i\theta} q_k$

$\Rightarrow$ depends only on shape

Geometric interpretation

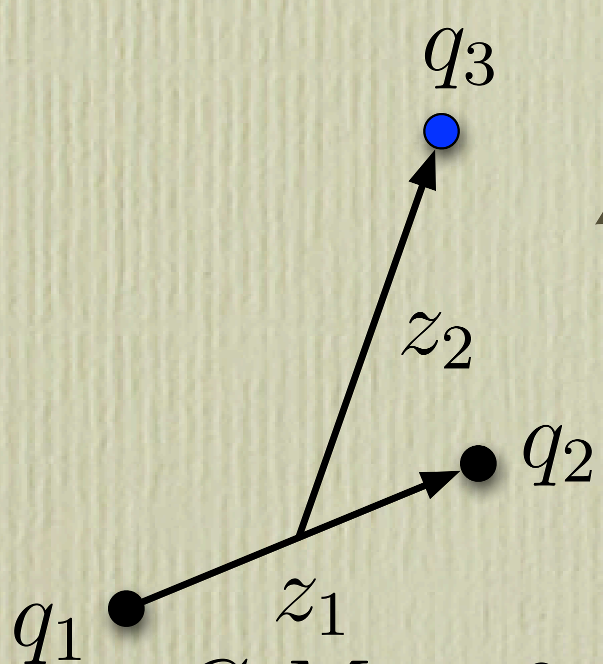
$$\zeta = \frac{z_2}{z_1}$$

$$\xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{1}{2} - \frac{\zeta}{3},$$

$$\xi_2 = \frac{q_2}{q_2 - q_1} = +\frac{1}{2} - \frac{\zeta}{3},$$

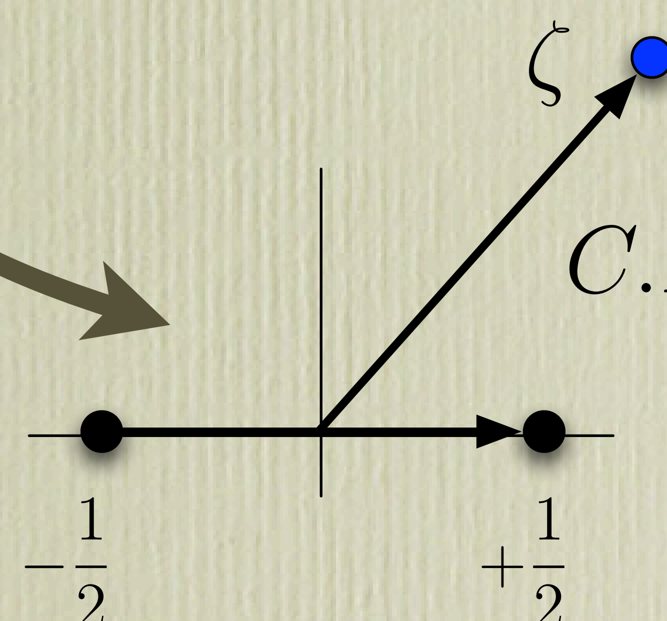
$$\xi_3 = \frac{q_3}{q_2 - q_1} = \frac{2\zeta}{3}.$$

$C.M. = 0$



$C.M. = 0$

rotation,
scaling
&
translation



$C.M. = \frac{\zeta}{3}$

Lagrangian

$$q_k = \sqrt{I} e^{i\theta} \frac{\xi_k}{\sqrt{\sum |\xi_\ell|^2}} \quad \Rightarrow \; L = \frac{1}{2} K + \frac{1}{\alpha} U$$

$$\frac{K}{2} = \frac{1}{2} \sum |\dot{q}_k|^2 = \frac{\dot{I}^2}{8I} + \frac{I}{2} \left(\dot{\theta} + \frac{\frac{2}{3}\zeta \wedge \dot{\zeta}}{\frac{1}{2} + \frac{2}{3}|\zeta|^2} \right)^2 + \frac{I}{6} \frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2},$$

$$U = \sum \frac{1}{|q_i - q_j|^\alpha} = \frac{\mu(\zeta)}{I^{\alpha/2}},$$

$$\mu = \left(\frac{1}{2} + \frac{2}{3}|\zeta|^2 \right)^{\alpha/2} \left(1 + \frac{1}{|\zeta - \frac{1}{2}|^\alpha} + \frac{1}{|\zeta + \frac{1}{2}|^\alpha} \right).$$

Angular momentum & size

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow C = \frac{\partial L}{\partial \dot{\theta}} = I\left(\dot{\theta} + \frac{\frac{2}{3}\zeta \wedge \dot{\zeta}}{\frac{1}{2} + \frac{2}{3}|\zeta|^2}\right) = \text{ constant}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{I}}\right) = \frac{\partial L}{\partial I}$$ by a few lines calculations, we get

$$\frac{d}{dt}\left(\frac{I^2}{6}\frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2}\right) = \frac{I^{1-\alpha/2}}{\alpha}\frac{d\mu}{dt}$$: Saari's relation

I times kinetic energy for the shape motion

$$\Rightarrow \frac{d}{ds}\left(\frac{1}{6}\left|\frac{d\zeta}{ds}\right|^2\right) = \frac{I^{1-\alpha/2}}{\alpha}\frac{d\mu}{ds}, \quad \frac{d}{ds} = \frac{I}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)}\frac{d}{dt}$$

Equation of motion for the Shape variable

$$\zeta = x + iy \in \mathbb{C} \sim \mathbf{x} = (x, y) \in \mathbb{R}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \frac{\partial L}{\partial \mathbf{x}}$$

↓

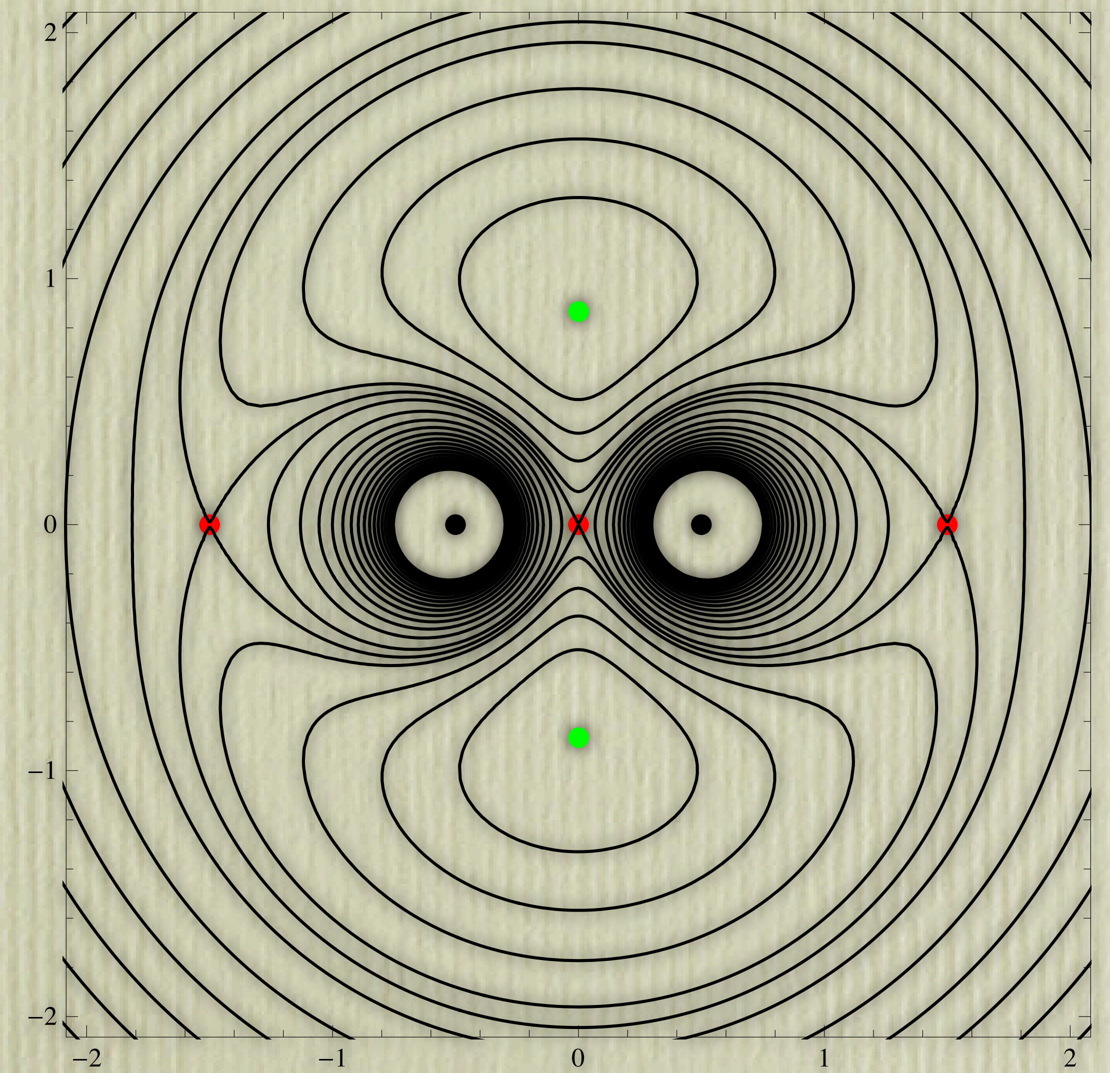
$$\frac{d^2\mathbf{x}}{ds^2} = \frac{2C - \frac{4}{3}\left(\mathbf{x} \wedge \frac{d\mathbf{x}}{ds}\right)}{\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2}\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) + \frac{3I^{1-\alpha/2}}{\alpha}\frac{\partial \mu}{\partial \mathbf{x}}.$$

Saari's conjecture for planar equal mass 3-body problem

$$m_k = 1, \quad k = 1, 2, 3,$$

$$U = \sum \frac{1}{|q_i - q_j|}$$

Saari's conjecture



$$\frac{d\mu(\mathbf{x})}{dt} = 0 \Leftrightarrow \frac{d\mathbf{x}}{dt} = 0.$$

Lagrange & Euler solutions

contours for $\mu(\mathbf{x}) = \text{constant}$

$\zeta = x + iy$: shape variable

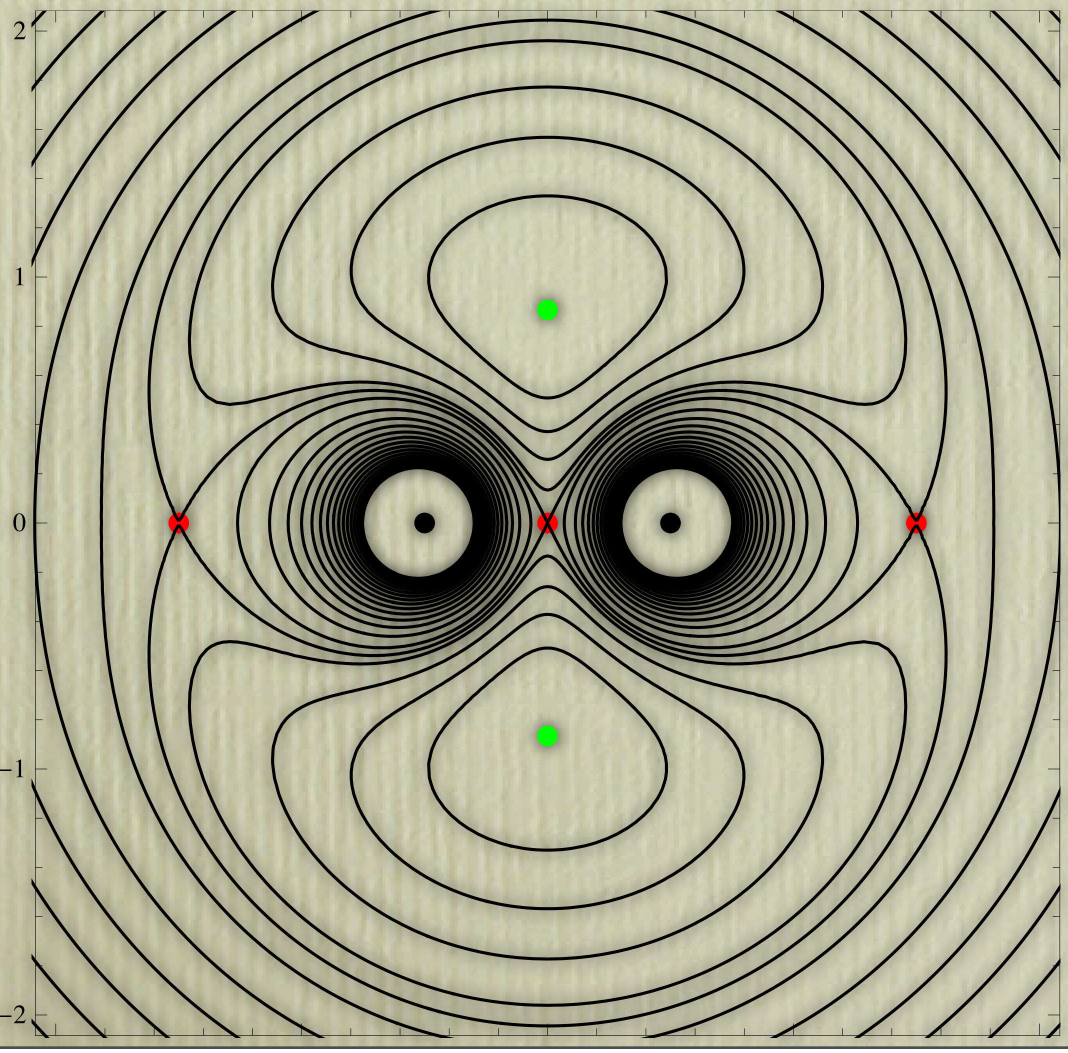
$$\mu = \sqrt{\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3}} \left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}} \right)$$

$$= \sqrt{\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2} \left(1 + \frac{1}{\sqrt{(x-1/2)^2 + y^2}} + \frac{1}{\sqrt{(x+1/2)^2 + y^2}} \right)$$

If $\dfrac{d\mu(\mathbf{x})}{ds} = 0$

Saari's relation: $$\frac{d}{ds}\left(\frac{1}{6}\left|\frac{d\mathbf{x}}{ds}\right|^2\right) = \sqrt{I}\frac{d\mu}{ds} = 0$$

$$\left|\frac{d\mathbf{x}}{ds}\right| = v \text{ : constant}$$



Saari's conjecture: $v = 0$

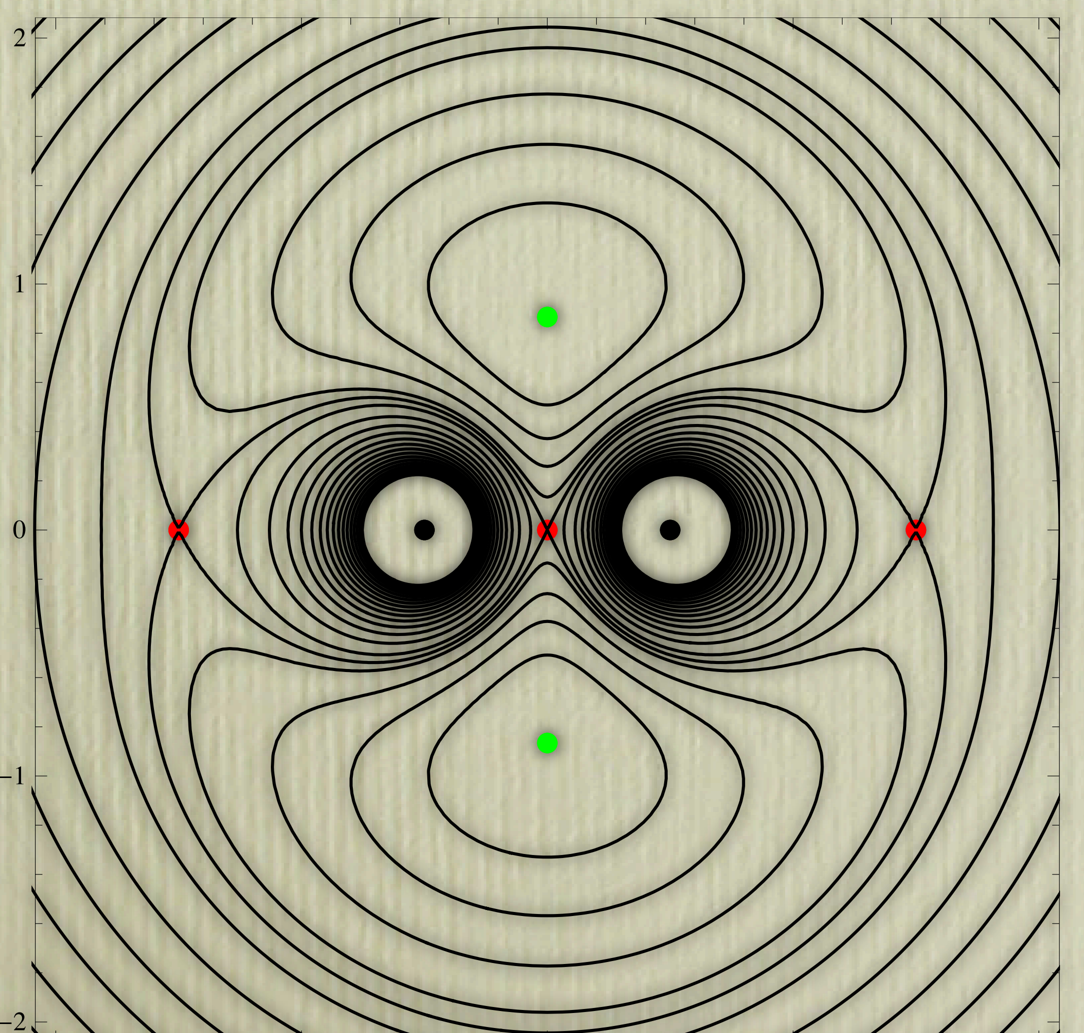
We assume $v > 0$

$$\frac{d\mu(\mathbf{x})}{ds}=0\ ,\ \left|\frac{d\mathbf{x}}{ds}\right|=v\ \Rightarrow\ \frac{d\mathbf{x}}{ds}=\frac{\epsilon v}{|\partial\mu/\partial\mathbf{x}|}\left(-\frac{\partial\mu}{\partial y},\frac{\partial\mu}{\partial x}\right)$$

$$\epsilon=\pm 1$$

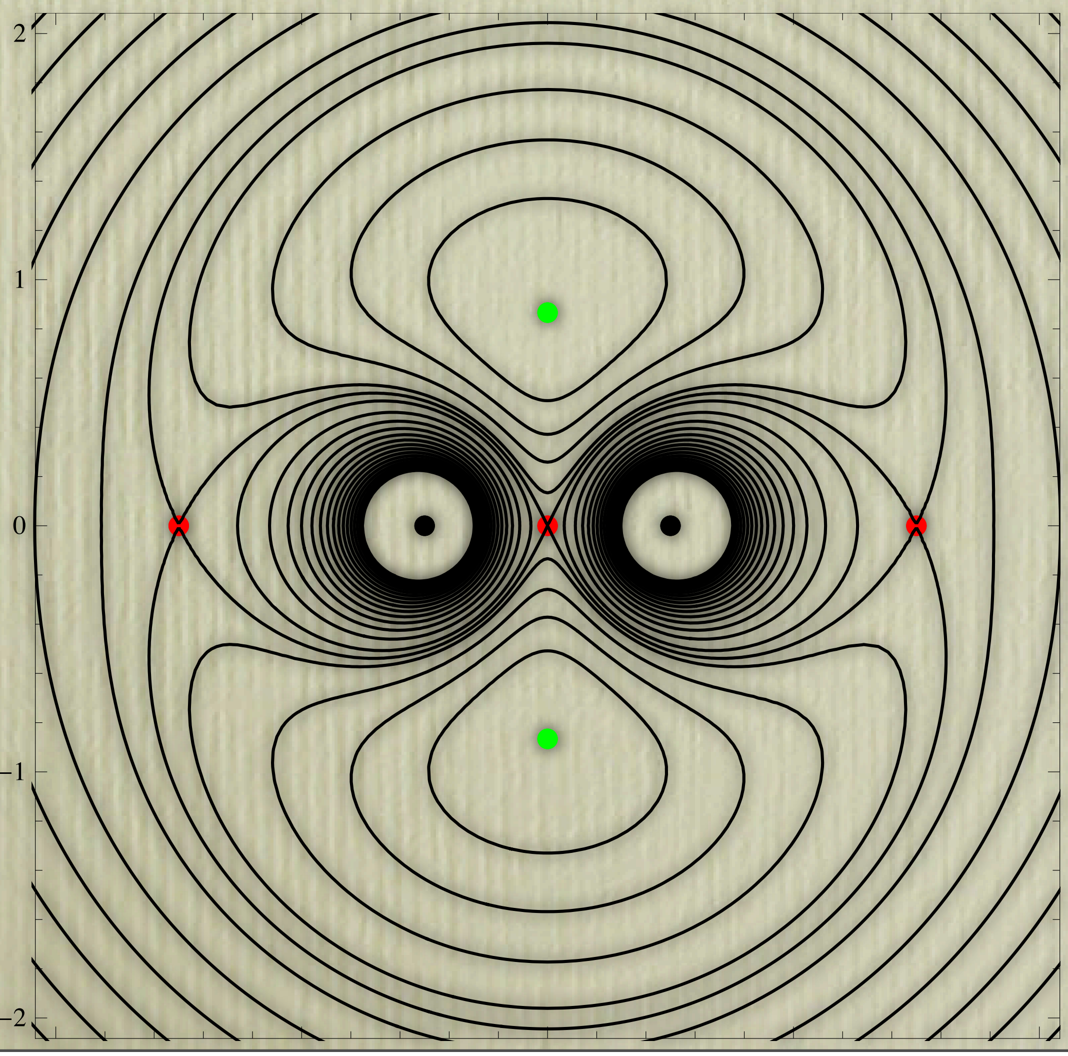
$\Rightarrow$ curvature of the orbit in $\mathbf{x}$ by the eq. of motion;

$$\kappa(\mathbf{x})=\frac{1}{1+4|\mathbf{x}|^2/3}\left(-\frac{4C}{v}+\frac{8\epsilon}{3|\partial\mu/\partial\mathbf{x}|}\left(\mathbf{x}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)\right)-\frac{3\epsilon\sqrt{I}}{v^2}\left|\frac{\partial\mu}{\partial\mathbf{x}}\right|$$



On the other hand, the curve $\mu(\mathbf{x}) = \text{constant}$ has the curvature;

$$\kappa(\mathbf{x}) = \frac{\epsilon}{|\partial\mu/\partial\mathbf{x}|^3}\left(\left(\frac{\partial\mu}{\partial y}\right)^2\frac{\partial^2\mu}{\partial x^2} - 2\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y}\frac{\partial^2\mu}{\partial x\partial y} + \left(\frac{\partial\mu}{\partial x}\right)^2\frac{\partial^2\mu}{\partial y^2}\right)$$



Two expressions for the curvature must be equal,

$$\sqrt{I}$$

$$= -\frac{4\epsilon C v}{3(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} + \frac{8v^2}{9(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|^2}\left(\mathbf{x}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)$$

$$-\frac{v^2}{3|\partial\mu/\partial\mathbf{x}|^4}\left(\left(\frac{\partial\mu}{\partial y}\right)^2\frac{\partial^2\mu}{\partial x^2} - 2\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y}\frac{\partial^2\mu}{\partial x\partial y} + \left(\frac{\partial\mu}{\partial x}\right)^2\frac{\partial^2\mu}{\partial y^2}\right)$$

take $v=\sqrt{3}$ using the scale invariance

$$q_k \to \lambda q_k, \quad t \to \lambda^{3/2} t$$

$$\sqrt{I} \to \lambda\sqrt{I}, \quad C \to \lambda^{1/2} C, \quad v \to \lambda^{1/2} v$$

$$\sqrt{I}$$

$$= -\frac{4C}{\sqrt{3}(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} + \frac{8}{3(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|^2}\left(\mathbf{x}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)$$

$$-\frac{1}{|\partial\mu/\partial\mathbf{x}|^4}\left(\left(\frac{\partial\mu}{\partial y}\right)^2\frac{\partial^2\mu}{\partial x^2} - 2\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y}\frac{\partial^2\mu}{\partial x\partial y} + \left(\frac{\partial\mu}{\partial x}\right)^2\frac{\partial^2\mu}{\partial y^2}\right)$$

iSummary いちどまとめます

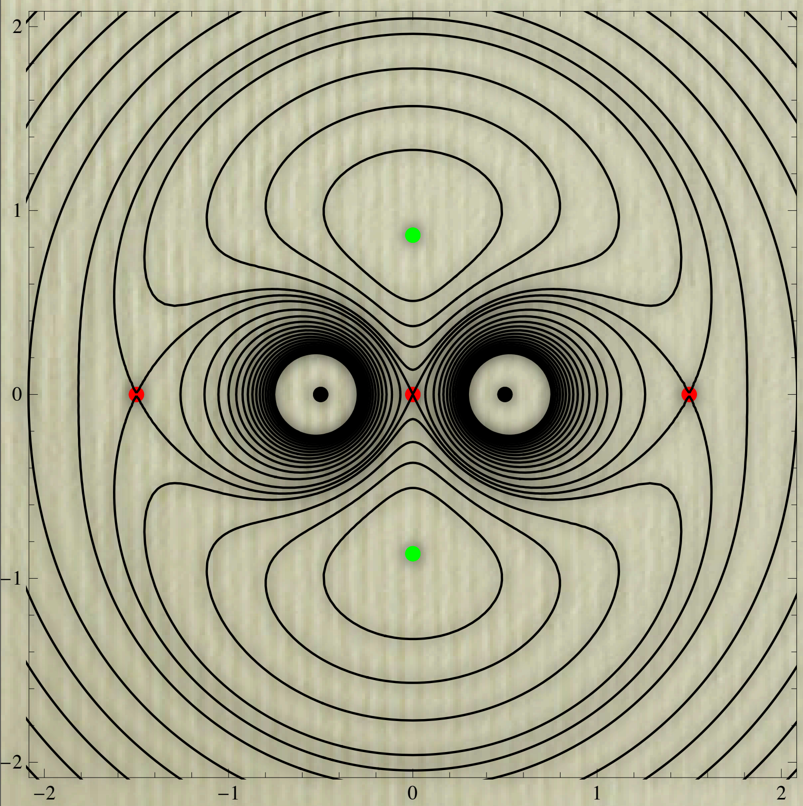
$\zeta = x + iy \sim \mathbf{x} = (x, y)$: shape variable

$$\frac{d\mu}{dt} = 0 \quad \Rightarrow \quad \frac{I^2 \left|\frac{d\mathbf{x}}{dt}\right|^2}{\left(\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2\right)^2} = \left|\frac{d\mathbf{x}}{ds}\right|^2 = v^2 = \text{ const.}$$

For $v > 0$:

$$\frac{d\mathbf{x}}{ds} = \frac{\epsilon v}{|\partial\mu/\partial\mathbf{x}|}\left(-\frac{\partial\mu}{\partial y}, \frac{\partial\mu}{\partial x}\right)$$ & eq. of motion

determines the curvature.



While, the curve $\mu =$ constant has its own cuvature.

a Big Problem is ...

$$\sqrt{I}$$

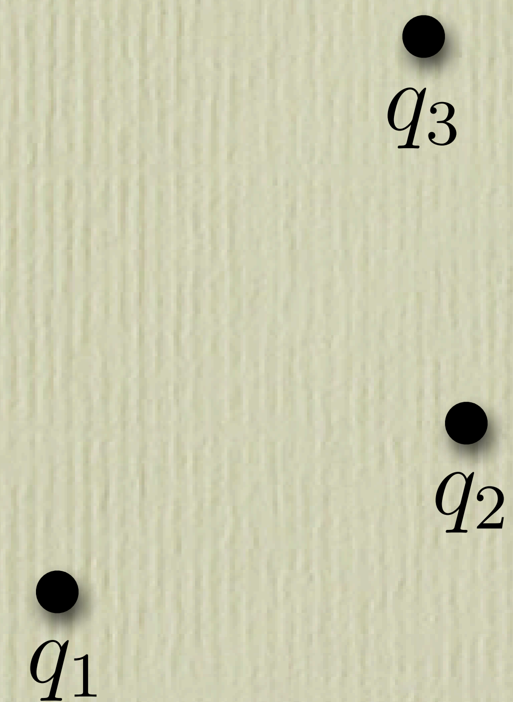
$$= -\frac{4C}{\sqrt{3}(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} + \frac{8}{3(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|^2}\left(\mathbf{x}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)$$

$$-\frac{1}{|\partial\mu/\partial\mathbf{x}|^4}\left(\left(\frac{\partial\mu}{\partial y}\right)^2\frac{\partial^2\mu}{\partial x^2} - 2\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y}\frac{\partial^2\mu}{\partial x\partial y} + \left(\frac{\partial\mu}{\partial x}\right)^2\frac{\partial^2\mu}{\partial y^2}\right)$$

This equation is too complex to treat.

Can we find a concise expression?

Symmetry & Invariants



$$\zeta = \frac{3}{2}\frac{q_3}{q_2 - q_1}$$

equal mass $\Rightarrow$ the system is invariant for $q_i \leftrightarrow q_j$

Invariants I

the configurational measure

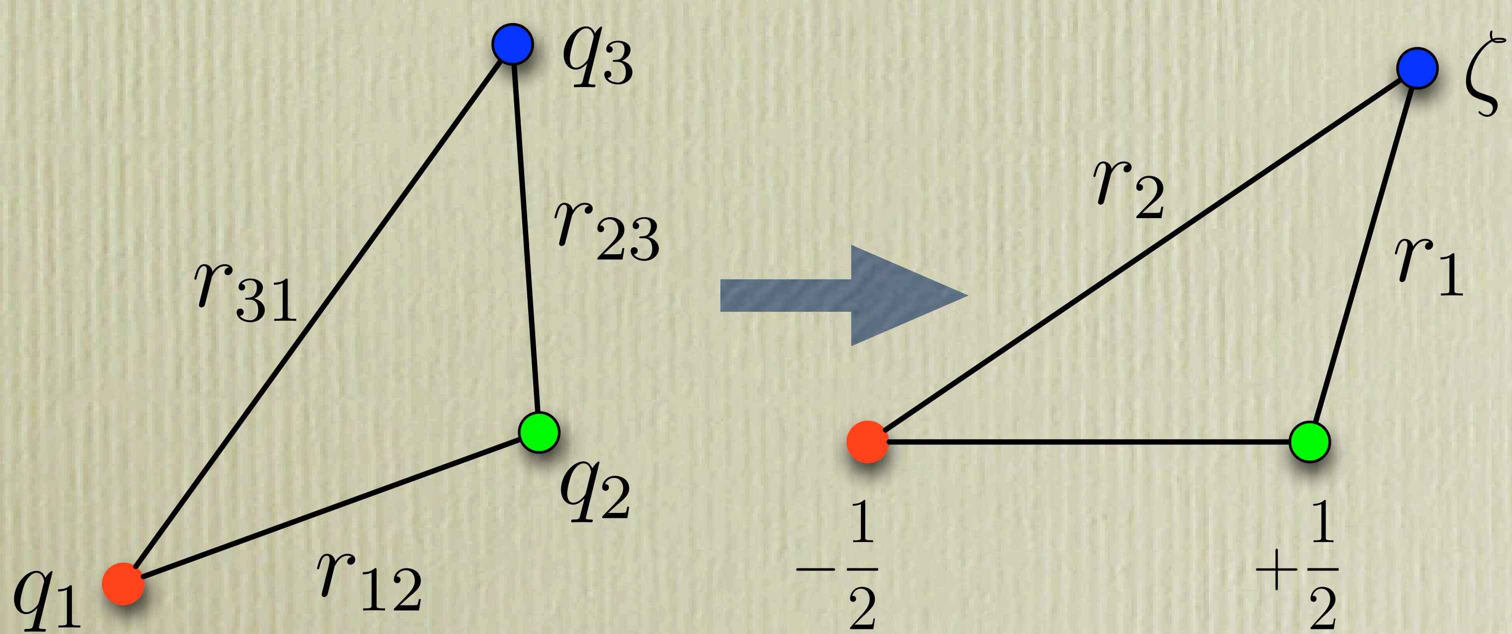
$$\mu = \sqrt{\frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)}\left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}}\right)$$

$$= \sqrt{\frac{1}{3}(1 + r_1^2 + r_2^2)}\left(1 + \frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$= \mu_3 + \mu_1 + \mu_2$$

$$r_1 = r_{23}/r_{12},$$
$$r_2 = r_{31}/r_{12}$$

$$\mu_3 = \sqrt{\frac{1}{3}(1 + r_1^2 + r_2^2)},$$
$$\mu_1 = \mu_3/r_1,$$
$$\mu_2 = \mu_3/r_2$$
$$\Updownarrow$$
$$r_1 = \mu_3/\mu_1,$$
$$r_2 = \mu_3/\mu_2$$



Invariants I

the configurational measure

$$\mu = \sqrt{\frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)}\left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}}\right)$$

$$= \sqrt{\frac{1}{3}(1 + r_1^2 + r_2^2)}\left(1 + \frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$= \mu_3 + \mu_1 + \mu_2$$

$$r_1 = r_{23}/r_{12},$$
$$r_2 = r_{31}/r_{12}$$

$$\mu_3 = \sqrt{\frac{1}{3}(1 + r_1^2 + r_2^2)},$$
$$\mu_1 = \mu_3/r_1,$$
$$\mu_2 = \mu_3/r_2$$
$$\Updownarrow$$
$$r_1 = \mu_3/\mu_1,$$
$$r_2 = \mu_3/\mu_2$$

is obviously invariant under

$$(q_1 \leftrightarrow q_2) \Rightarrow (\mu_1 \leftrightarrow \mu_2)$$

$$(q_1 \to q_2 \to q_3 \to q_1) \Rightarrow (\mu_1 \to \mu_2 \to \mu_3 \to \mu_1)$$

the same exchange rule

Invariants I

ratio of symmetric polynomials of μ_1, μ_2, μ_3.

if

$$f(r_1, r_2) + g(r_1, r_2)\sqrt{\frac{1 + r_1^2 + r_2^2}{3}} = f\left(\frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2}\right) + g\left(\frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2}\right)\mu_3$$

is invariant under the exchange $q_i \leftrightarrow q_j$,

then it is invariant under the exchange $\mu_i \leftrightarrow \mu_j$, and

it must have manifestly invariant form $h(\mu, \nu, \rho)$.

here, f, g and h are rational functions and

$$\mu = \mu_1 + \mu_2 + \mu_3,$$

$$\nu = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = \sqrt{2\mu\rho + 3\rho^2},$$

$$\rho = \mu_1\mu_2\mu_3$$

elementary symmetric polynomials of μ_1, μ_2, μ_3.

12年1月7日土曜日

sample

$$\left(\mu = \sqrt{\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3}} \left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}}\right)\right)$$

$$\begin{aligned}
&= \sqrt{\frac{1 + r_1^2 + r_2^2}{3}} \left(1 + \frac{1}{r_1} + \frac{1}{r_2}\right) \\
&= \mu_3 \left(1 + \frac{1}{\mu_3/\mu_1} + \frac{1}{\mu_3/\mu_2}\right) \\
&= \mu_1 + \mu_2 + \mu_3 \\
&= \mu.
\end{aligned}$$

Invariant function must have

manifestly invariant expression

$$\Rightarrow \sqrt{I} = \frac{4C}{\ldots} + \ldots$$

Shape variables

So, we use 3 kind of shape variables

$$\zeta = x + iy$$

$$r_1, r_2$$

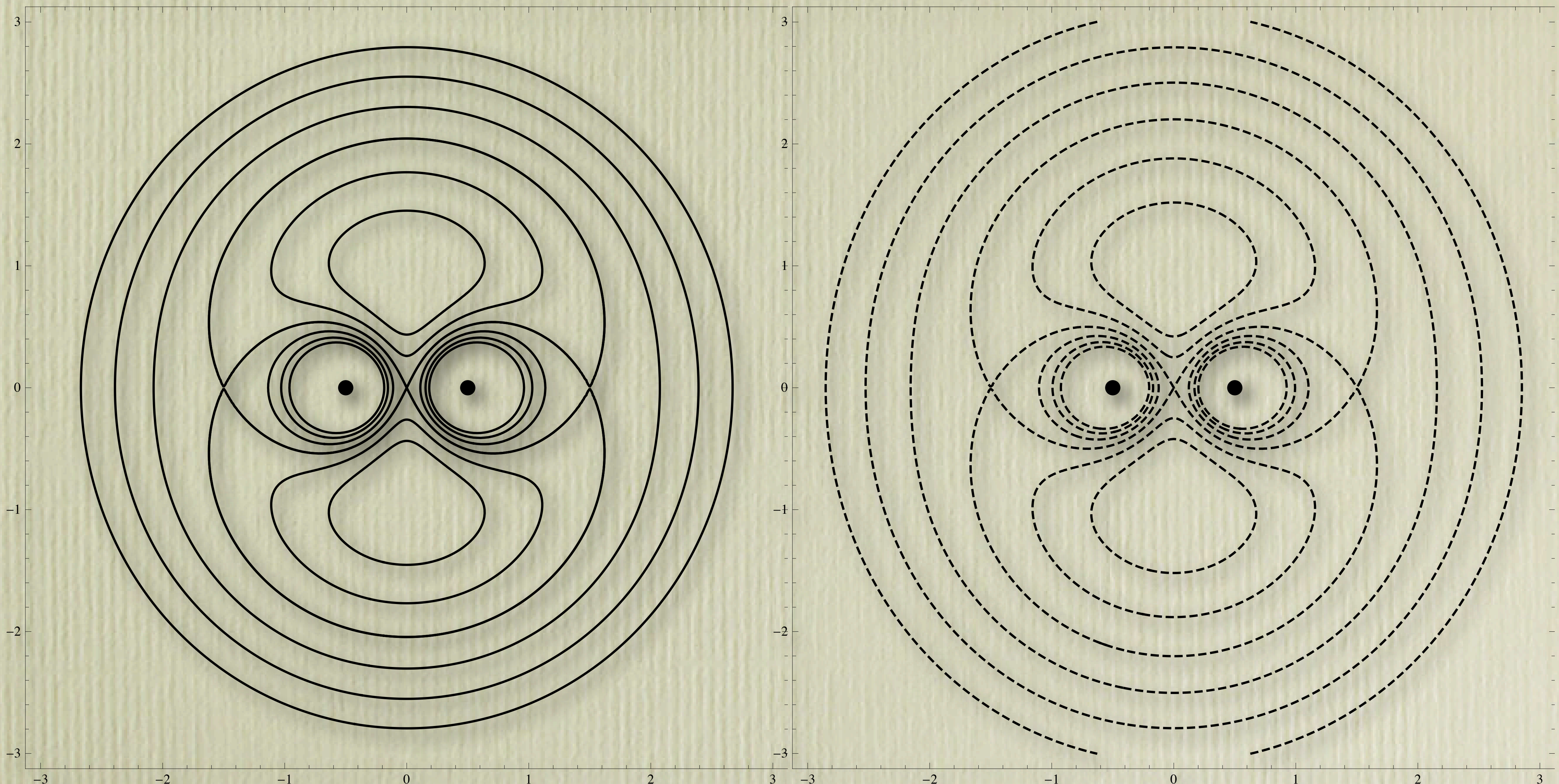
$$\mu = \mu_1 + \mu_2 + \mu_3,$$

$$\nu = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = \sqrt{2\mu\rho + 3\rho^2},$$

$$\rho = \mu_1\mu_2\mu_3$$

μ and ρ are manifestly invariant shape variables

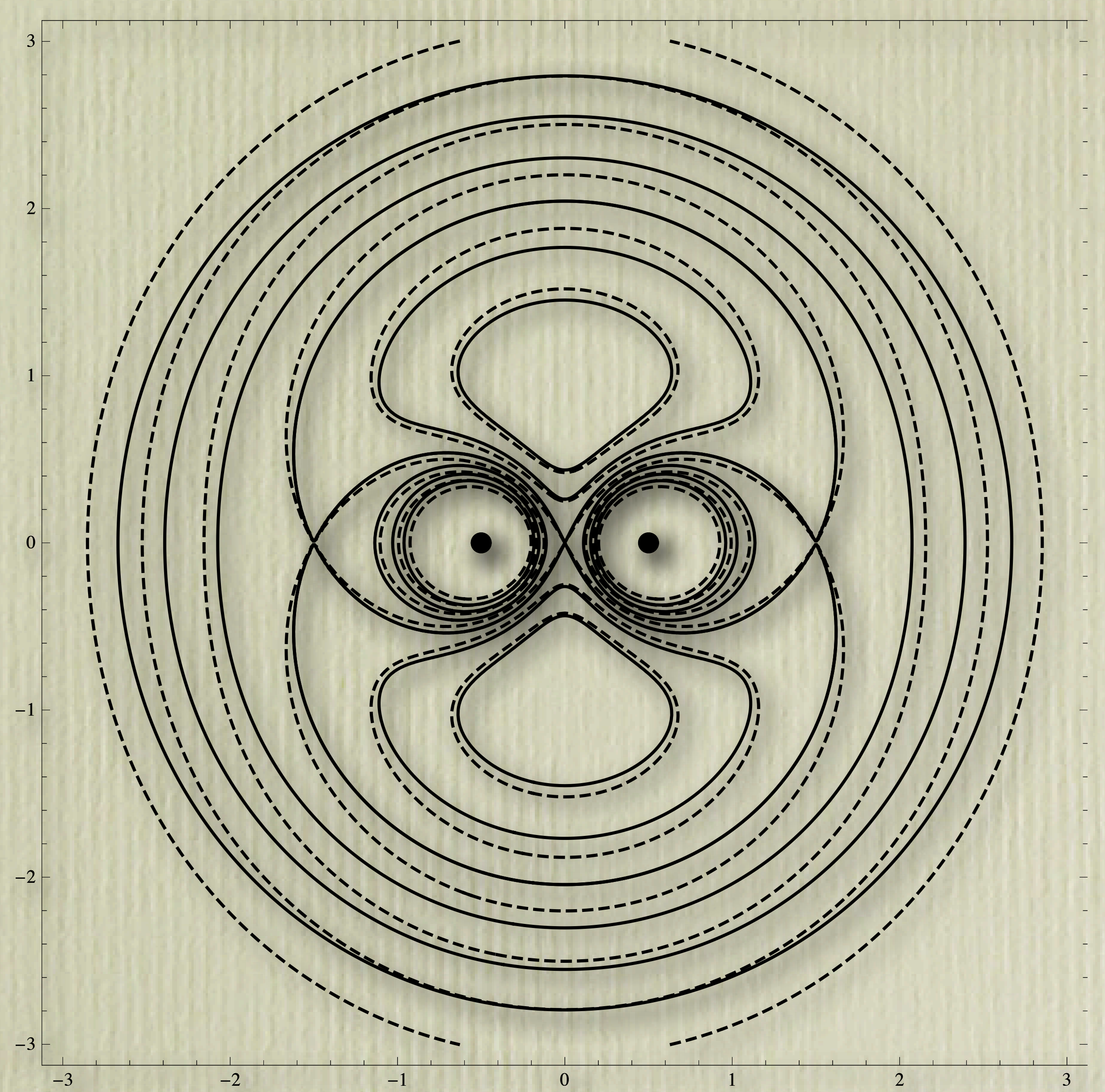
Invariant shape variables



μ ρ

Contour plots for $\mu = \mu_1 + \mu_2 + \mu_3$ and $\rho = \mu_1 \mu_2 \mu_3$.

Invariant shape variables



Contour plots for $\mu = \mu_1 + \mu_2 + \mu_3$ and $\rho = \mu_1\mu_2\mu_3$.

Invariant II

$$\zeta = \frac{3}{2}\frac{q_3}{q_2 - q_1}$$

$$(q_1 \leftrightarrow q_2) \Rightarrow \zeta \to -\zeta$$

$$(q_1 \to q_2 \to q_3 \to q_1) \Rightarrow \zeta \to \zeta' = \frac{3}{2}\frac{q_1}{q_3 - q_2} = \frac{1}{2}\frac{3/2 + \zeta}{1/2 - \zeta}$$

Direct calculation shows that
the kinetic energy for the shape change is invariant;

$$\frac{1}{3}\frac{|d\zeta|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2}$$

$\Rightarrow$ Metric space: $g_{\alpha\beta} = \dfrac{1}{3}\dfrac{\delta_{\alpha\beta}}{\left(\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2\right)^2}$

$$d\mathbf{x} = (dx^1, dx^2), \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha}$$

Invariants II-1

$$
\begin{aligned}
|\nabla\mu|^2 &= \sum g^{\alpha\beta}(\partial_\alpha\mu)(\partial_\beta\mu) \\
&= \frac{3}{4}\left(1+\frac{4}{3}|\mathbf{x}|^2\right)^2\left(\left(\frac{\partial\mu}{\partial x}\right)^2+\left(\frac{\partial\mu}{\partial y}\right)^2\right) \\
&= \frac{1}{3}(1+r_1^2+r_2^2)^2\left(\left(\frac{\partial\mu}{\partial r_1}\right)^2+\left(\frac{\partial\mu}{\partial r_2}\right)^2+\frac{r_1^2+r_2^2-1}{r_1r_2}\frac{\partial\mu}{\partial r_1}\frac{\partial\mu}{\partial r_2}\right) \\
&= \frac{1+r_1^2+r_2^2}{9r_1^4r_2^4}\Bigg(2r_1^4r_2^4(r_1^2+r_2^2) \\
&\qquad + r_1^4r_2^4(r_1+r_2)-r_1r_2(r_1^7+r_2^7)-r_1^4r_2^4-4r_1^3r_2^3(r_1+r_2) \\
&\qquad + (2r_1^6+r_1^5r_2-r_1^4r_2^2-4r_1^3r_2^3-r_1^2r_2^4+r_1r_2^5+2r_2^6) \\
&\qquad + r_1r_2(r_1^3+r_2^3)+2(r_1^4+r_2^4)-r_1r_2\Bigg).
\end{aligned}
$$

Substituting $r_1=\mu_3/\mu_1$ and $r_2=\mu_3/\mu_2$, we get ...

Invariants II-1

$$|\nabla\mu|^2 = \frac{(\mu_1^2\mu_2^2+\mu_2^2\mu_3^2+\mu_3^2\mu_1^2)}{9\mu_1^6\mu_2^6\mu_3^6}\Bigg(-(\mu_1^7\mu_2^7+\mu_2^7\mu_3^7+\mu_3^7\mu_1^7)$$
$$-\mu_1^4\mu_2^4\mu_3^4(\mu_1^2+\mu_2^2+\mu_3^2)-4\mu_1^4\mu_2^4\mu_3^4(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)$$
$$+2(\mu_1^8\mu_2^4\mu_3^2+\dots)+(\mu_1^7\mu_2^4\mu_3^3+\dots)\Bigg).$$

Symmetric polynomials of μ_1, μ_2, μ_3

This can be expressed by
elementary symmetric polynomials

$$\mu = \mu_1+\mu_2+\mu_3,$$
$$\nu = \mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1,$$
$$\rho = \mu_1\mu_2\mu_3$$

Invariants II-1

$$|\nabla \mu|^2 = -\mu^2 + 2\mu^4 + 6\mu\rho - 9\rho^2 - 3(2\mu^2 - \mu\rho + 3\rho^2)\sqrt{2\mu\rho + 3\rho^2}.$$

Here, we eliminated ν using $\nu = \sqrt{2\mu\rho + 3\rho^2}$.

Invariants II-2

$$\Delta\mu = \sum_{ij} \frac{1}{\sqrt{|g|}} \partial_i \left(g^{ij} \sqrt{|g|} \partial_j \mu \right)$$

$$= \frac{3}{4} \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \mu$$

$$= \mu + 2\mu^3 + 6\rho - 6\mu\sqrt{2\mu\rho + 3\rho^2}$$

Invariants II-3

$$\lambda = \sum_{ij} g^{ij}(\partial_i \mu)(\partial_j |\nabla \mu|^2)$$

$$= \frac{3}{4}\left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \left(\frac{\partial \mu}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \mu}{\partial y}\frac{\partial}{\partial y}\right)|\nabla \mu|^2$$

$$= \frac{1}{2}\Bigg(4\mu^3 - 24\mu^5 + 32\mu^7 - 72\mu^2\rho + 660\mu^4\rho + 324\mu\rho^2$$

$$+36\mu^3\rho^2 - 432\rho^3 + 891\mu^2\rho^3 + 2349\mu\rho^4 - 243\rho^5$$

$$+\Big(24\mu^3 - 60\mu^5 - 156\mu^2\rho + 28\mu^4\rho + 324\mu\rho^2$$

$$-93\mu^3\rho^2 - 216\rho^3 - 27\mu^2\rho^3 + 81\mu\rho^4\Big)\sqrt{2\mu\rho + 3\rho^2}\Bigg).$$

12年1月7日土曜日

Manifestly invariant form

$$\sqrt{I} = -\frac{2C}{|\nabla\mu|} + \frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}.$$

The right hand side is expressed by manifestly invariant variables μ and ρ.

The time dependent variable is only ρ.

The Saari's conjecture

easy as pie

$$\sqrt{I} = -\frac{2C}{|\nabla\mu|} + \frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}$$

$$= a_0 + a_{1/2}\sqrt{\rho} + a_1\rho + O(\rho^{3/2}).$$

$$a_0 = \frac{2(1-\mu^2 + C\sqrt{-1+2\mu^2})}{\mu(1-2\mu^2)},$$

$$a_{1/2} = \frac{3\sqrt{2}\big((-2+\mu^2)\sqrt{-1+2\mu^2} - 2C(-1+2\mu^2)\big)}{(1-2\mu^2)^2\sqrt{\mu(-1+2\mu^2)}},$$

$$a_1 = \frac{3\big((-2+\mu^2)(1+6\mu^2) - 2C(1+7\mu^2)\sqrt{-1+2\mu^2}\big)}{\mu^2(-1+2\mu^2)^3},$$

The Saari's conjecture

$$\sqrt{I} = -\frac{2C}{|\nabla\mu|} + \frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}$$

$$= a_0 + a_{1/2}\sqrt{\rho} + a_1\rho + O(\rho^{3/2}).$$

$$\left|\frac{d\mathbf{x}}{ds}\right|^2 = v^2 = 3$$

$$\frac{1}{2}\left(\frac{d\sqrt{I}}{dt}\right)^2 = \frac{9\left(\mu^7\left(C\left(2-4\mu^2\right)+\left(-2+\mu^2\right)\sqrt{-1+2\mu^2}\right)^2\right)}{16\left(\left(1-2\mu^2\right)^2\left(1-\mu^2+C\sqrt{-1+2\mu^2}\right)^4\right)}\rho + O(\rho^{3/2})$$

$$\Rightarrow\ E = \frac{1}{2}\left(\frac{d\sqrt{I}}{dt}\right)^2 + \frac{C^2+1}{2I} - \frac{\mu}{\sqrt{I}}$$: must be identity

$$= e_0(C,\mu) + e_{1/2}(C,\mu)\sqrt{\rho} + e_1(C,\mu)\rho + O(\rho^{3/2}).$$

The Saari's conjecture

$$e_{1/2}(C,\mu) = 0 \;\Rightarrow\; C = -\frac{1}{\sqrt{-1+2\mu^2}},\; \frac{-2+\mu^2}{2\sqrt{-1+2\mu^2}}.$$

For $C = -\dfrac{1}{\sqrt{-1+2\mu^2}} \;\Rightarrow\; e_1 = -\dfrac{9\mu(-2+\mu^2)}{16(-1+2\mu^2)} < 0$

For $C = \dfrac{-2+\mu^2}{2\sqrt{-1+2\mu^2}} \;\Rightarrow\; e_1 = \dfrac{3\mu(-2+\mu^2)}{4(-1+2\mu^2)} > 0$

$$\mu = \sqrt{\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3}} \left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}} \right) \geq 3$$

The Saari's conjecture

Therefore,

$$E = \frac{1}{2}\left(\frac{d\sqrt{I}}{dt}\right)^2 + \frac{C^2+1}{2I} - \frac{\mu}{\sqrt{I}}$$

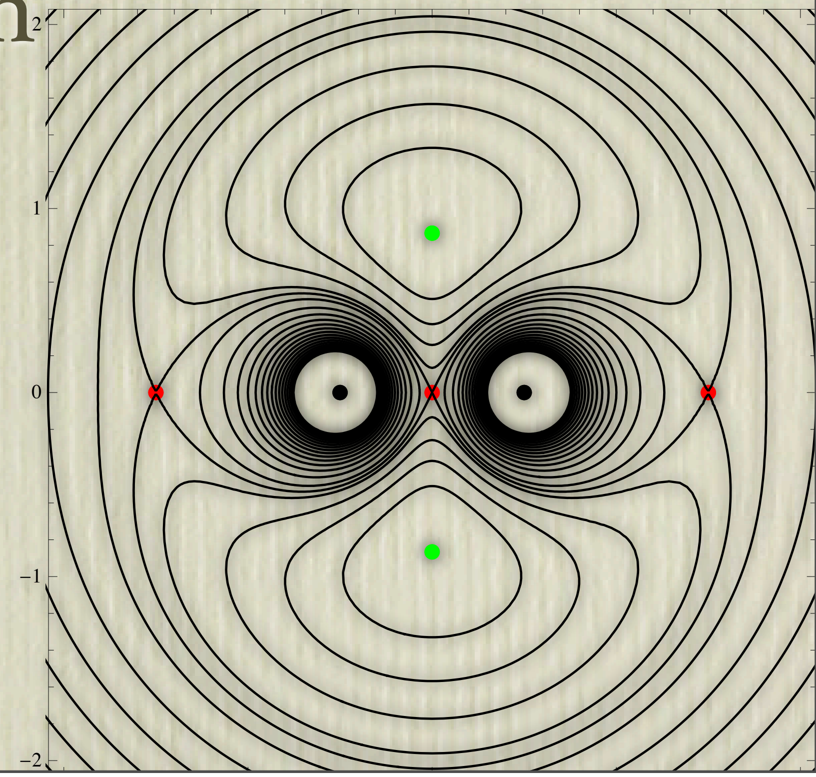
is not satisfied by all ρ.

Namely, there is NO finite orbit with

$$\frac{d\mu}{dt} = 0 \text{ and } \frac{d\mathbf{x}}{dt} \neq 0.$$

So, we proved the Saari's conjecture

$$\frac{d\mu}{dt} = 0 \Rightarrow \frac{d\mathbf{x}}{dt} = 0.$$



The surface ... ?

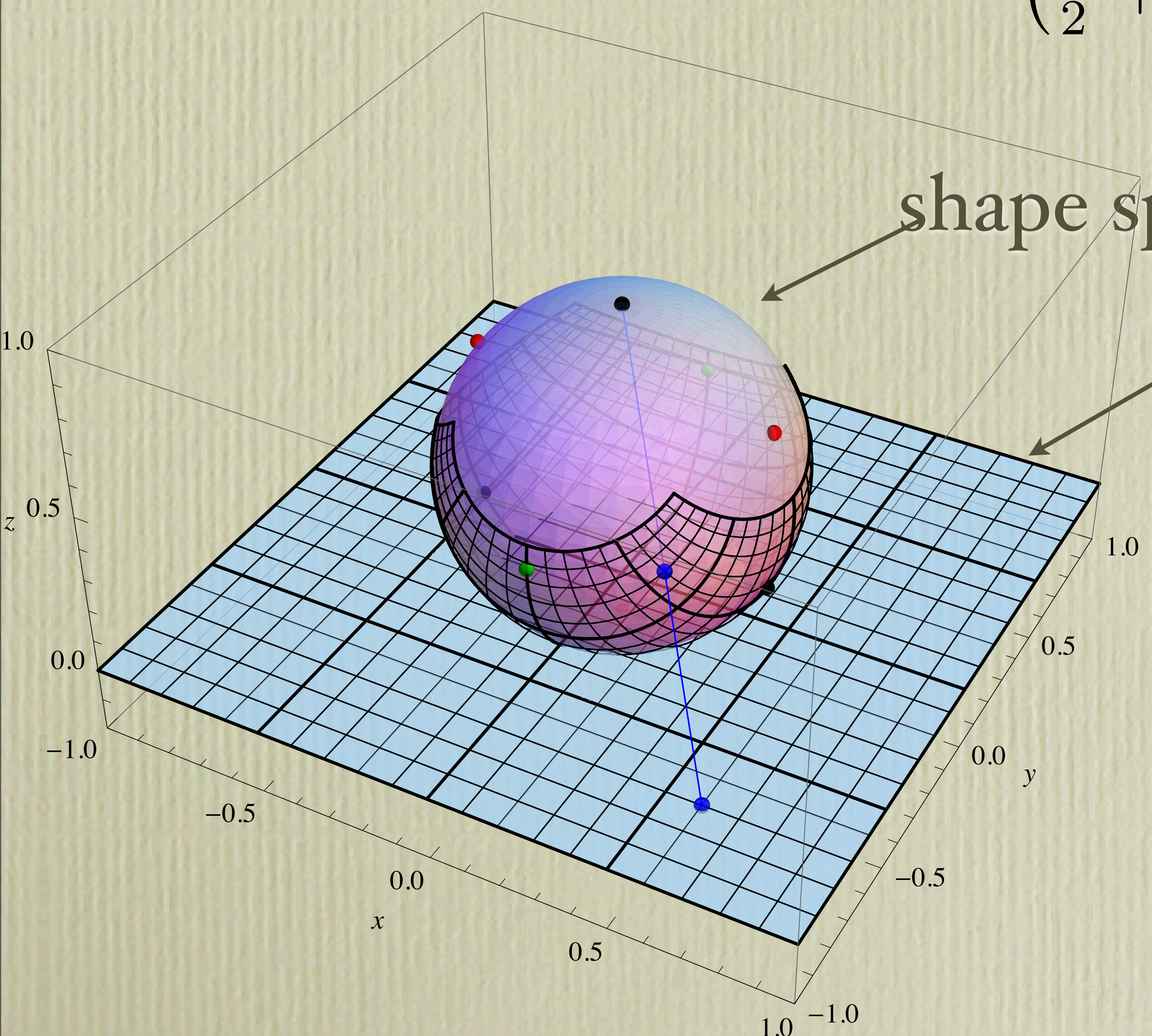
$$g_{\alpha\beta} = \frac{1}{3} \frac{\delta_{\alpha\beta}}{\left(\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2\right)^2}$$

The Shape Sphere

$$g_{\alpha\beta} = \frac{1}{3} \frac{\delta_{\alpha\beta}}{\left(\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2\right)^2}$$

shape sphere: the Riemann sphere

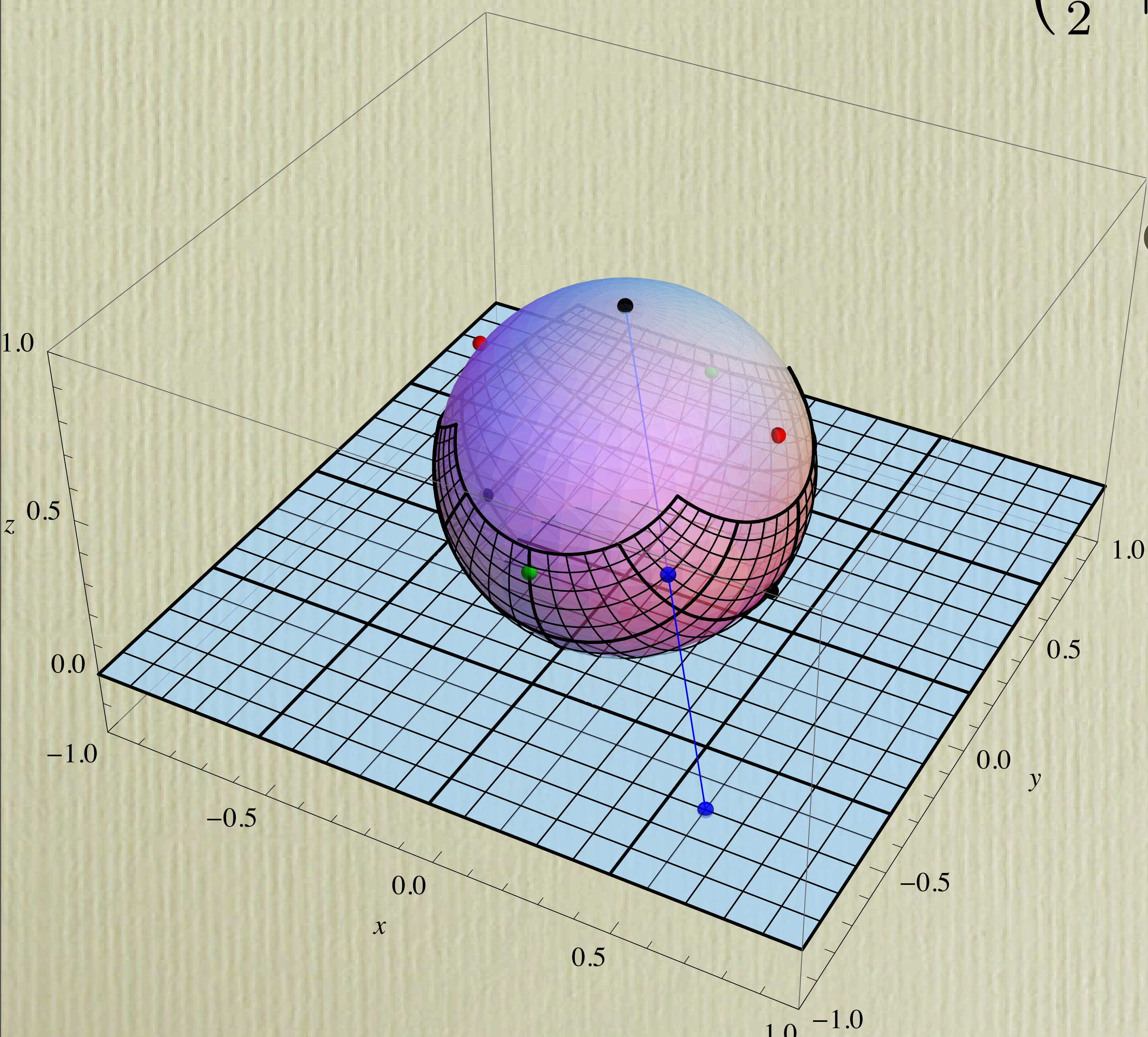
shape variable: $\zeta = x^1 + ix^2$



W. Hsiang 1995, 2006,
R. Montgomery 2002,
R. Moeckel 2007,
K. H. Kuwabara
& K. Tanikawa 2007

The Shape Sphere

$$g_{\alpha\beta} = \frac{1}{3} \frac{\delta_{\alpha\beta}}{\left(\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2\right)^2}$$

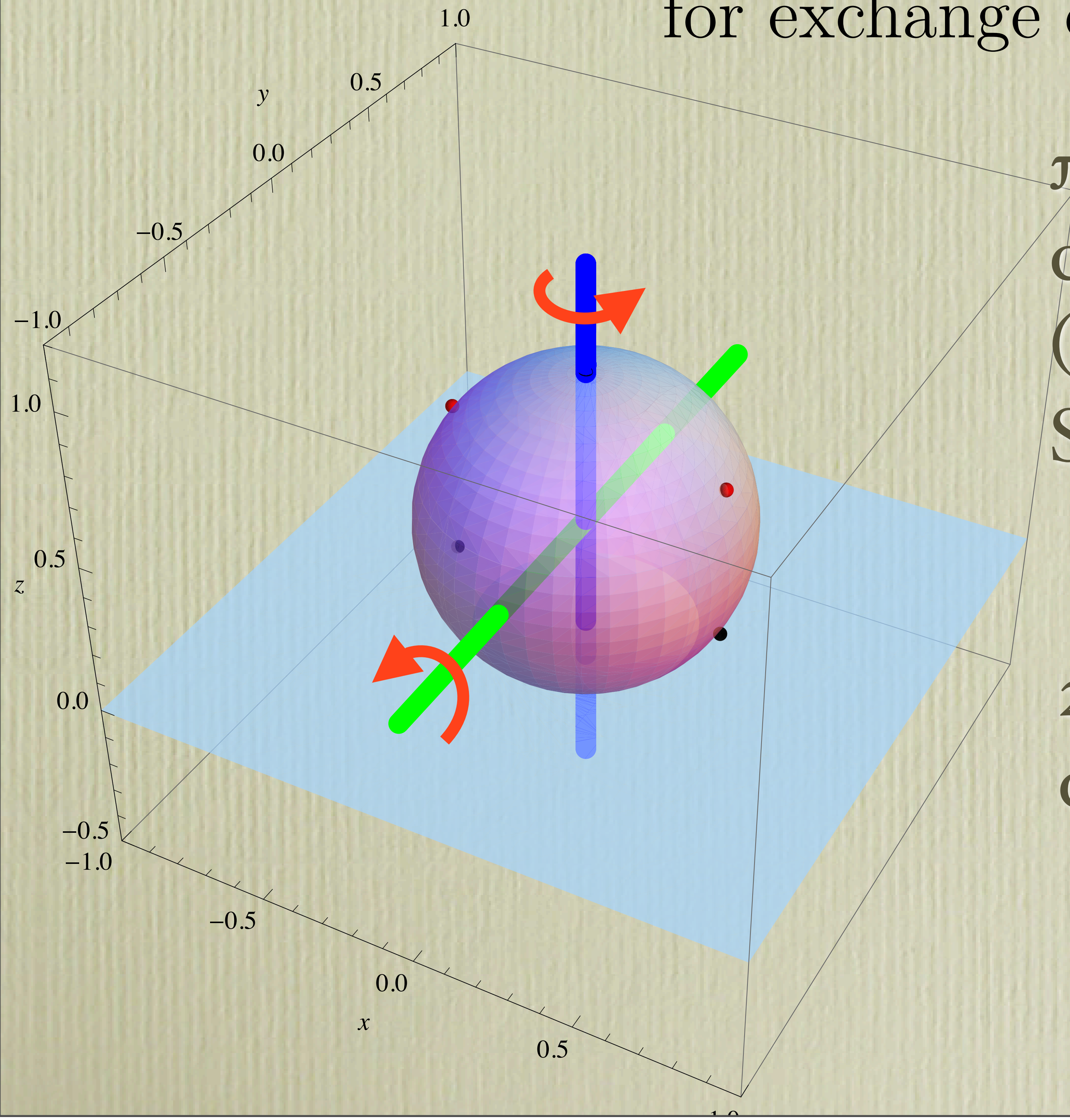


distance on the sphere:

$$\frac{1}{3} \frac{|d\zeta|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2}$$

The Shape Sphere

for exchange of $q_i \leftrightarrow q_j$



π rotation around **the line** connecting the North pole (2-body collision) and the South pole(Euler config.):

$$\zeta \to -\zeta$$

2π/3 rotation around **the line** connecting Lagrange points:

$$\zeta \to \zeta' = \frac{1}{2}\,\frac{3/2+\zeta}{1/2-\zeta}$$

Summary

Shape variables:

$$\zeta = \frac{3}{2}\frac{q_3}{q_2 - q_1}$$

$$r_1 = \frac{r_{23}}{r_{12}},\ r_2 = \frac{r_{31}}{r_{12}}$$

$$\mu = \left(\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3}\right)^{1/2}\left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}}\right),$$

$$\rho = \left(\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3}\right)^{3/2}\left(\frac{1}{r_{12}r_{23}r_{31}}\right)$$

Saari's conjecture:

$$\frac{d\mu}{dt} = 0 \text{ and } \frac{d\mathbf{x}}{dt} \neq 0 \Rightarrow \sqrt{I} = -\frac{2C}{|\nabla\mu|} + \frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}.$$

$$\Rightarrow E \neq \frac{1}{2}\left(\frac{d\sqrt{I}}{dt}\right)^2 + \frac{C^2+1}{2I} - \frac{\mu}{\sqrt{I}}$$

$$\therefore \frac{d\mu}{dt} = 0 \Rightarrow \frac{d\mathbf{x}}{dt} = 0$$

ありがとうございました