Decomposition of matrix for second derivative of action at choreographic three-body solutions

T. Fujiwara
with H. Fukuda, H. Ozaki
2017/06/07

2017 RIMS Workshop
Dynamical Systems — Joint Exploration of Theory and Application

Hessian

Equal mass planar three-body problem

$$L = \frac{1}{2}\sum_{\ell}\left(\frac{dq_\ell}{dt}\right)^2 + \sum_{i\neq j} V(|q_i - q_j|)$$

$$S[q+\delta q] = S[q] + \frac{1}{2}\int_0^T dt \sum_{i,j}\delta q_i \underbrace{\left(-\delta_{ij}\frac{d^2}{dt^2} + \frac{\partial^2}{\partial q_i \partial q_j}\sum V\right)}_{=H}\delta q_j$$

Eigenvalue problem at a critical point $\delta S[q] = 0$

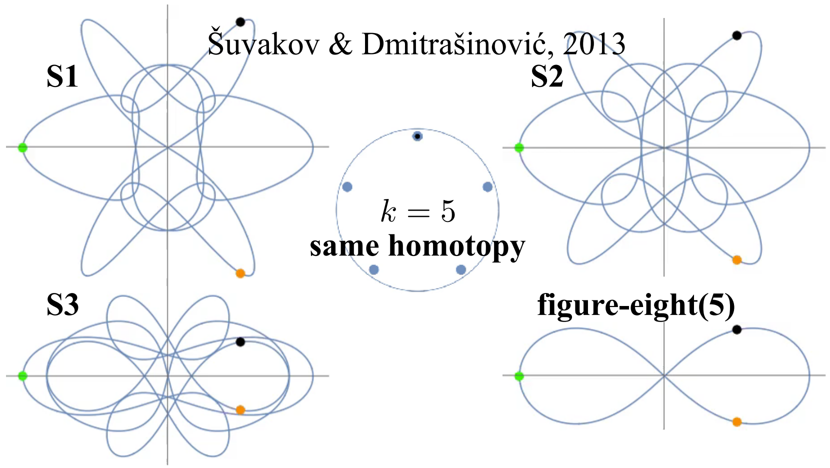
$$H\Psi = \lambda\Psi,\ \Psi = \begin{pmatrix}\delta q_0\\ \delta q_1\\ \delta q_2\end{pmatrix},\ \delta q_\ell = \begin{pmatrix}\delta q_{\ell x}\\ \delta q_{\ell y}\end{pmatrix} \in \mathbb{R}^2.$$

Numerical calculations of eigenvalues of Hessian at choreographic solutions

Mitsuru Shibayama
舞踏解に関する第二変分の数値計算
Computations and Calculations in Celestial Mechanics
Proceedings of Symposium on Celestial Mechanics and N-body Dynamics, 2010 Eds. M. Saito, M. Shibayama and M. Sekiguchi

Slalom solutions and figure-eight(5)

Šuvakov & Dmitrašinović, 2013



Slalom Solutions

- **Šuvakov, M.**

  ***Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere*, Celest. Mech. Dyn. Astron. 119, 369–377 (2014)**

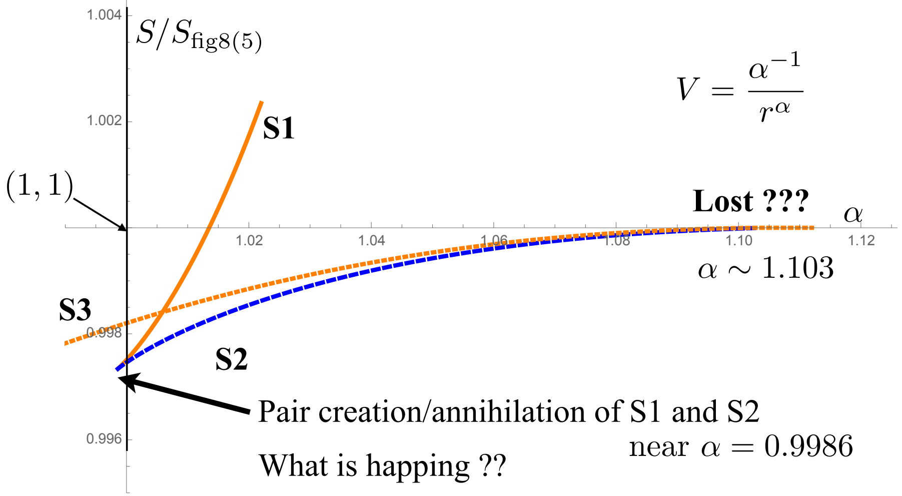
- **Šuvakov, M., Dmitrašinović, V.**

  ***Three classes of Newtonian three-body planar periodic orbits*, Phys. Rev. Lett. 110(11), 114301 (2013)**

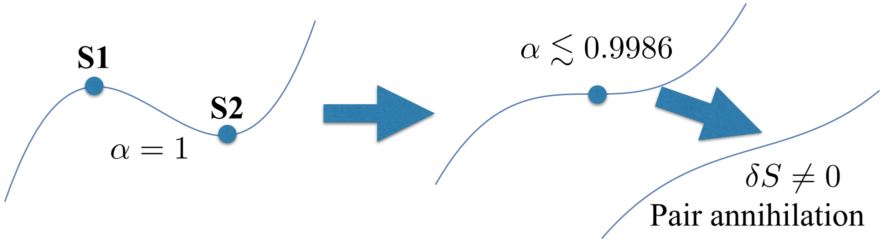
- **Šuvakov, M., Dmitrašinović, V.**

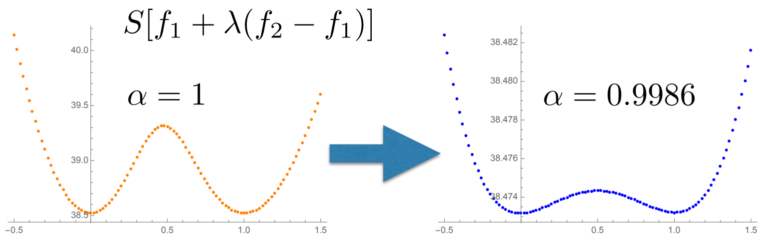
  ***A guide to hunting periodic three-body orbits*, Am. J. Phys. 82, 609–619 (2014)**

Trace the solutions



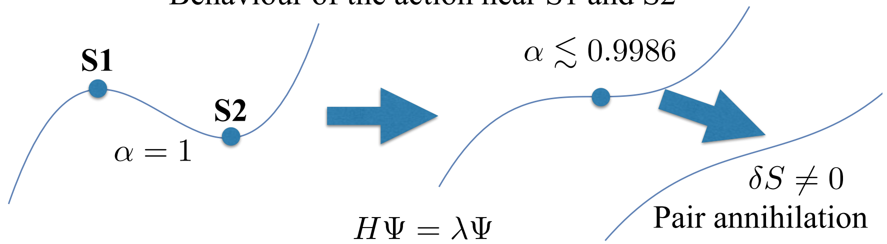
A scenario

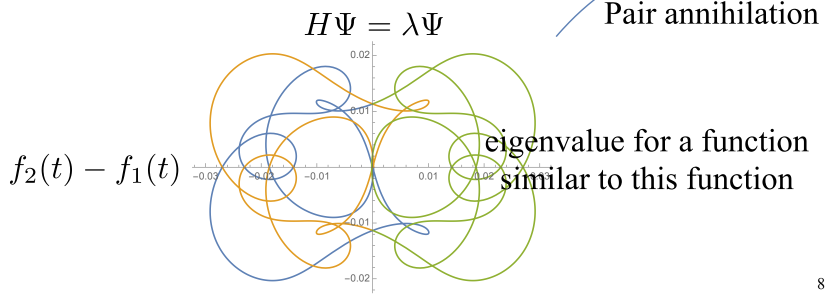




A scenario

Behaviour of the action near S1 and S2





Hessian

$$H = -\frac{d^2}{dt^2} + {}^t\Delta U \Delta$$

$$U = \begin{pmatrix} u_{12} & 0 & 0 \\ 0 & u_{20} & 0 \\ 0 & 0 & u_{01} \end{pmatrix}, \ \Delta = \begin{pmatrix} 0 & E_2 & -E_2 \\ -E_2 & 0 & E_2 \\ E_2 & -E_2 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \delta q_{0x} \\ \delta q_{0y} \\ \delta q_{1x} \\ \delta q_{1y} \\ \delta q_{2x} \\ \delta q_{2y} \end{pmatrix}, \ \begin{cases} \delta q_{\ell x} = \sum c^k_{\ell x} \cos(k\nu t) + \sum s^k_{\ell x} \sin(k\nu t) \\ \delta q_{\ell y} = \dots \end{cases}$$

$$\nu = \frac{2\pi}{T}$$

Hessian

particle	3
x, y	2
cos, sin	2
k	$\sim 3 \times 2^{10}$
	$\sim 36 \times 2^{10}$

$$H = \begin{pmatrix}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *
\end{pmatrix}$$

decompose

precision,
time, memory
to know properties of eigenfunction

Trivial modes

$$H = -\frac{d^2}{dt^2} + {}^t\Delta U \Delta$$

Difference operator

$$\Delta \begin{pmatrix} \delta q_0 \\ \delta q_1 \\ \delta q_2 \end{pmatrix} = \begin{pmatrix} \delta q_1 - \delta q_2 \\ \delta q_2 - \delta q_0 \\ \delta q_0 - \delta q_1 \end{pmatrix}$$

→ Trivial mode

$$\Delta \begin{pmatrix} \delta q \\ \delta q \\ \delta q \end{pmatrix} = 0$$

→ $$H\Psi = -\frac{d^2}{dt^2}\Psi, \qquad \lambda = \frac{4\pi^2}{T^2}k^2, \ k = 0, 1, 2, 3, \dots$$

$$\Delta = P - P^{-1}, \ P \begin{pmatrix} \delta q_0 \\ \delta q_1 \\ \delta q_2 \end{pmatrix} = \begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_0 \end{pmatrix}, \ P^3 = 1$$

Trivial and non-trivial modes

$$\lambda = \frac{4\pi^2}{T^2}k^2, \ k = 0, 1, 2, 3, \dots$$

trivial modes: quadruply degenerated

$\uparrow x, y, \cos, \sin$

$$H = \begin{pmatrix}
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & *
\end{pmatrix}$$

Choreographic symmetry

$$(q_0(t), q_1(t), q_2(t)) = (q(t), q(t+T/3), q(t-T/3)),\ q(t+T) = q(t)$$

Time shift $R(f(t)) = f(t+T/3)$

Cyclic permutation $P \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_0 \end{pmatrix}$

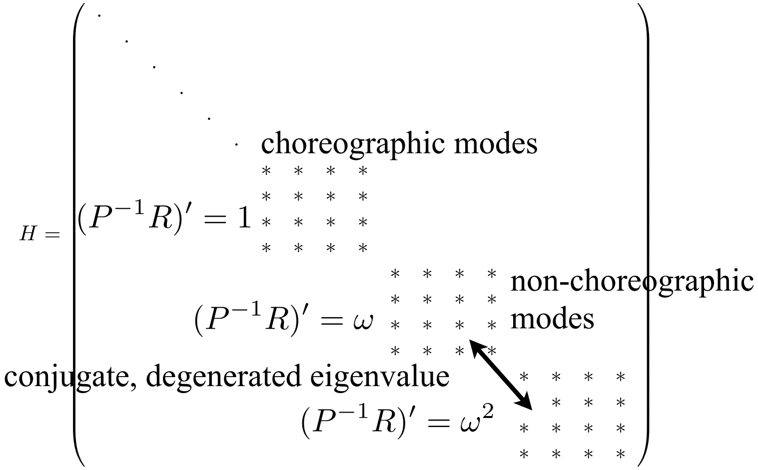
Choreographic solutions

$$R \begin{pmatrix} q(t) \\ q(t+T/3) \\ q(t-T/3) \end{pmatrix} = \begin{pmatrix} q(t+T/3) \\ q(t-T/3) \\ q(t) \end{pmatrix} = P \begin{pmatrix} q(t) \\ q(t+T/3) \\ q(t-T/3) \end{pmatrix}$$

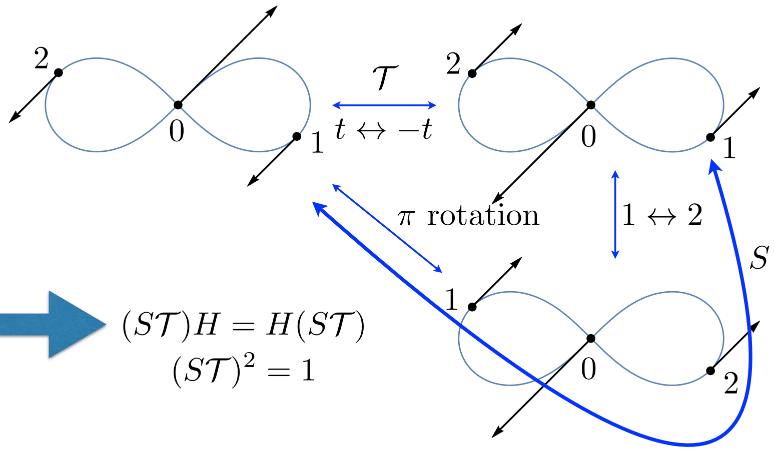
$$(P^{-1}R)H = H(P^{-1}R),\ (P^{-1}R)^3 = 1$$

$$\to (P^{-1}R)' = 1, \omega, \omega^2,\ \omega = \frac{-1+i\sqrt{3}}{2}$$

Choreographic and non-choreographic modes

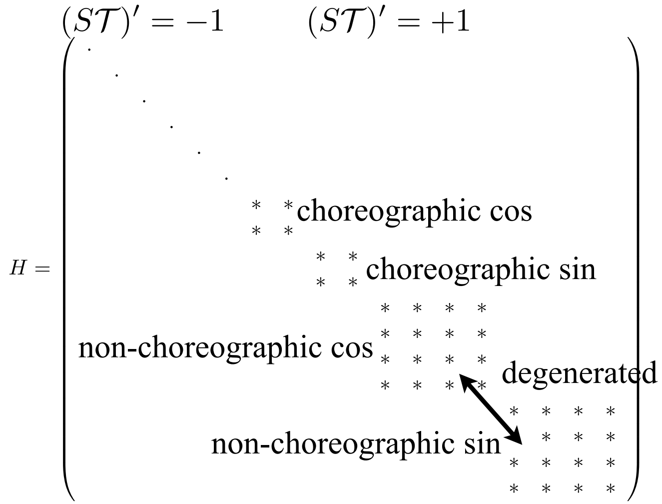


Time reversal symmetry for figure-eight family



$$(S\mathcal{T})H = H(S\mathcal{T})$$
$$(S\mathcal{T})^2 = 1$$

cos and sin mode



Hessian for choreographic sin mode

$$k_n = \frac{6n-(-1)^n-3}{4}, \quad n=1,2,3,\dots$$
$$= 1,2,4,5,7,8,10,11,\dots$$

positive integers that are not multiple of 3

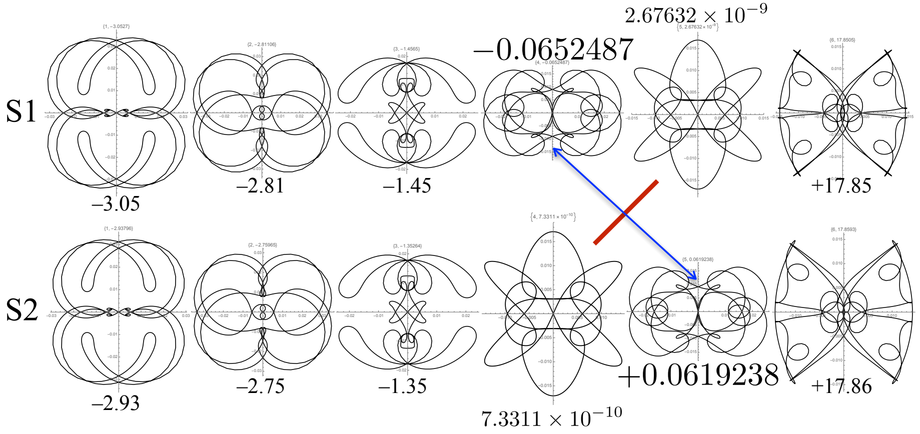
$$\Psi = \sum a_{nx}\begin{pmatrix}\sin(k_n\nu t)\\ 0\\ \sin(k_n\nu(t+T/3))\\ 0\\ \sin(k_n\nu(t-T/3))\\ 0\end{pmatrix} + \sum a_{ny}\begin{pmatrix}0\\ \sin(k_n\nu t)\\ 0\\ \sin(k_n\nu(t+T/3))\\ 0\\ \sin(k_n\nu(t-T/3))\end{pmatrix}$$

$$H_{mn} = \frac{4\pi^2}{T^2}k_n^2\delta_{mn}E_2 \qquad \nu = \frac{2\pi}{T}$$
$$+(-1)^{m+n}\frac{3}{T}\int_0^T dt\ u_{12}\Big(\cos((k_m-k_n)\nu t)+\cos((k_m+k_n)\nu t)\Big)$$

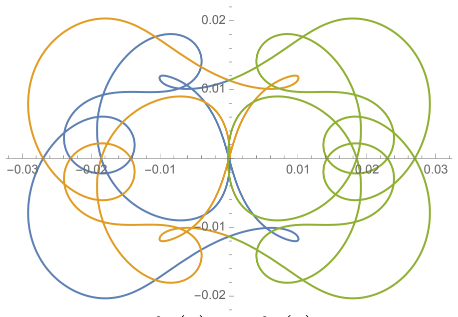
First 6 eigenvalues of S1 & S2

$\alpha = 0.9986$

choreographic sin modes $\lambda/(4\pi^2/T^2)$



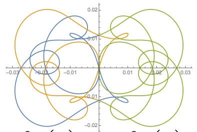
Difference of S1 and S2



$f_2(t) - f_1(t)$

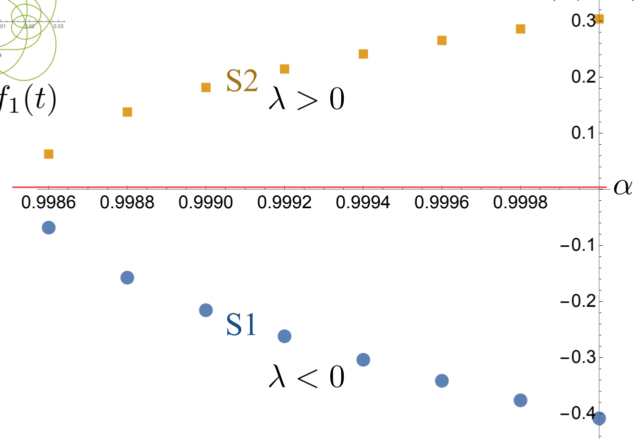
S2 S1

The eigenvalue of S1 & S2

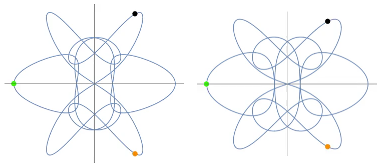


$f_2(t) - f_1(t)$

for $0.9986 \le \alpha \le 1$ $\lambda/(4\pi^2/T^2)$



summary



S1 & S2 at $\alpha = 1$