Figure-eight and Slalom solutions in function space

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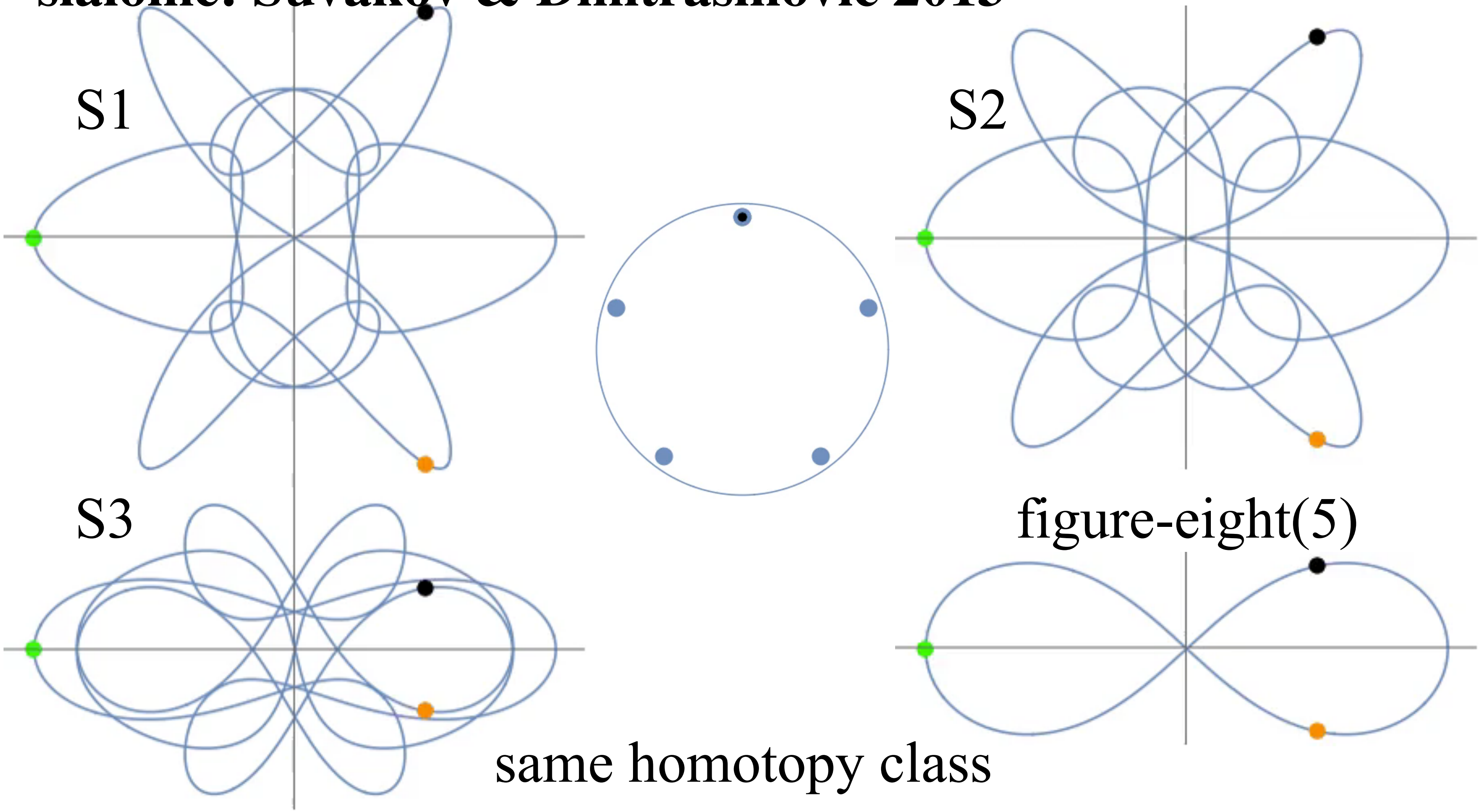
Figure-eight and slalom solutions

figure-eight: Moore 1993, Chenciner & Montgomery 2000

slalome: Šuvakov & Dmitrašinović 2013



same homotopy class

Three-body choreography

$$L = \frac{1}{2}\sum_k \left|\frac{dq_k}{dt}\right|^2 + \frac{1}{\alpha}\sum_{i,j}\frac{1}{|q_i - q_j|^\alpha}$$

$\alpha = 1$: Newton potential

$$q_0(t) = q(t), q_1(t) = q(t + T/3), q_2(t) = q(t + 2T/3)$$

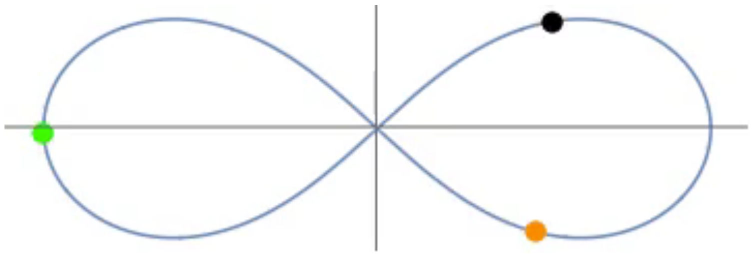
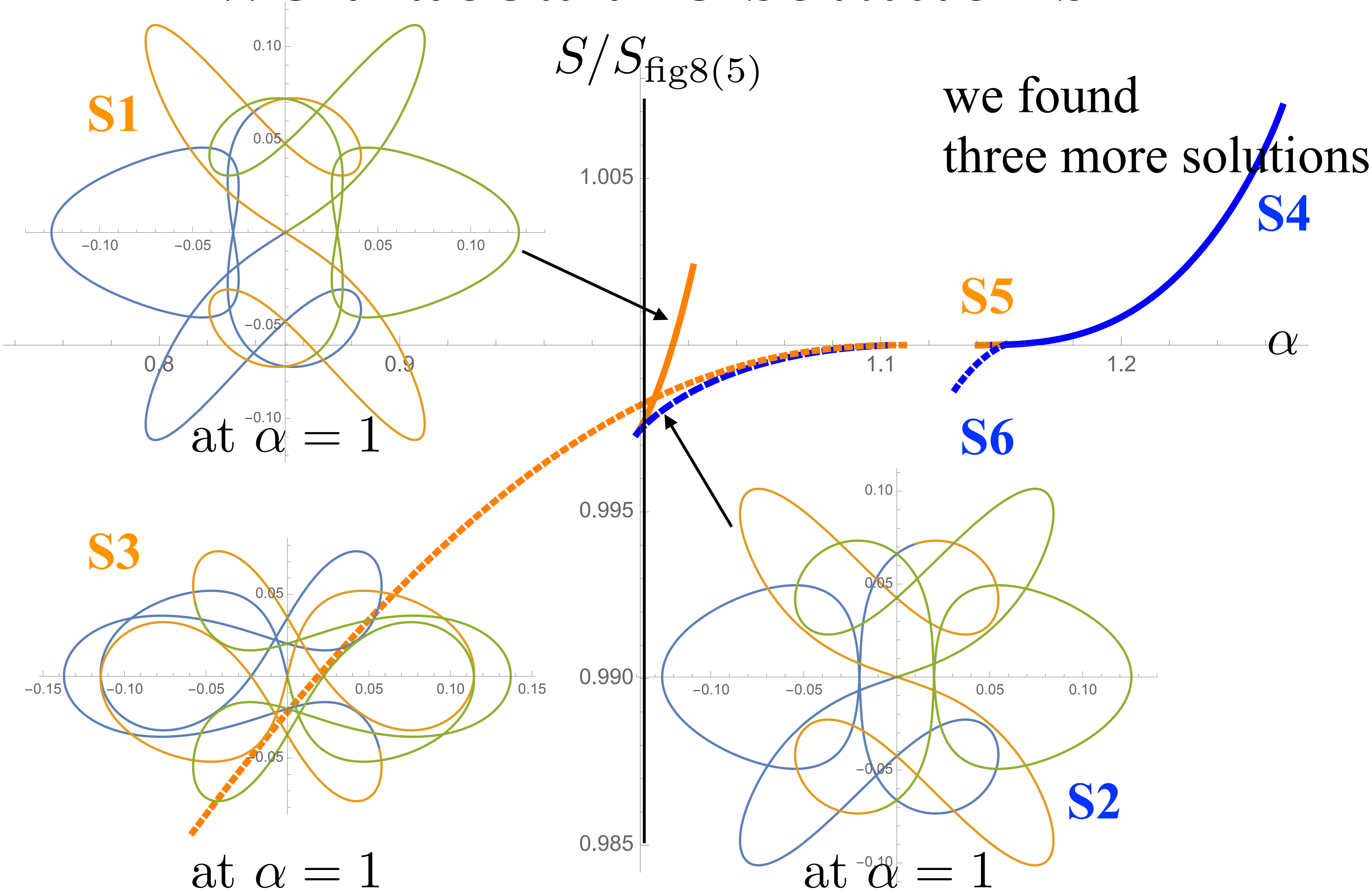


figure-eight solution
C. Moore 1993,
A. Chenciner and R. Montgomery 2000

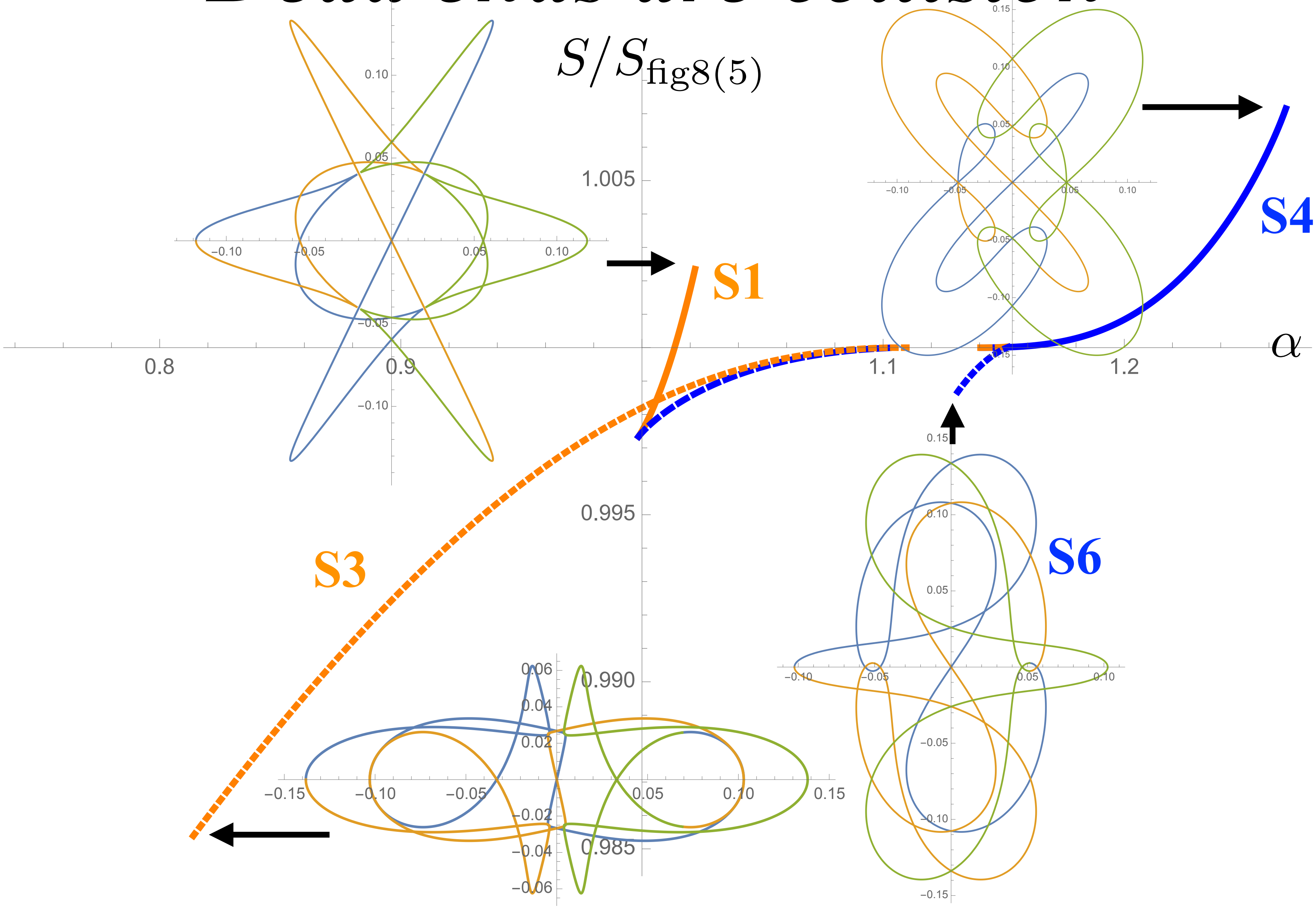
Slalom Solutions

- **Šuvakov, M.**
  - **Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere, Celest. Mech. Dyn. Astron. 119, 369–377 (2014)**
- **Šuvakov, M., Dmitrašinović, V.**
  - **Three classes of Newtonian three-body planar periodic orbits, Phys. Rev. Lett. 110(11), 114301 (2013)**
- **Šuvakov, M., Dmitrašinović, V.**
  - **A guide to hunting periodic three-body orbits, Am. J. Phys. 82, 609–619 (2014)**

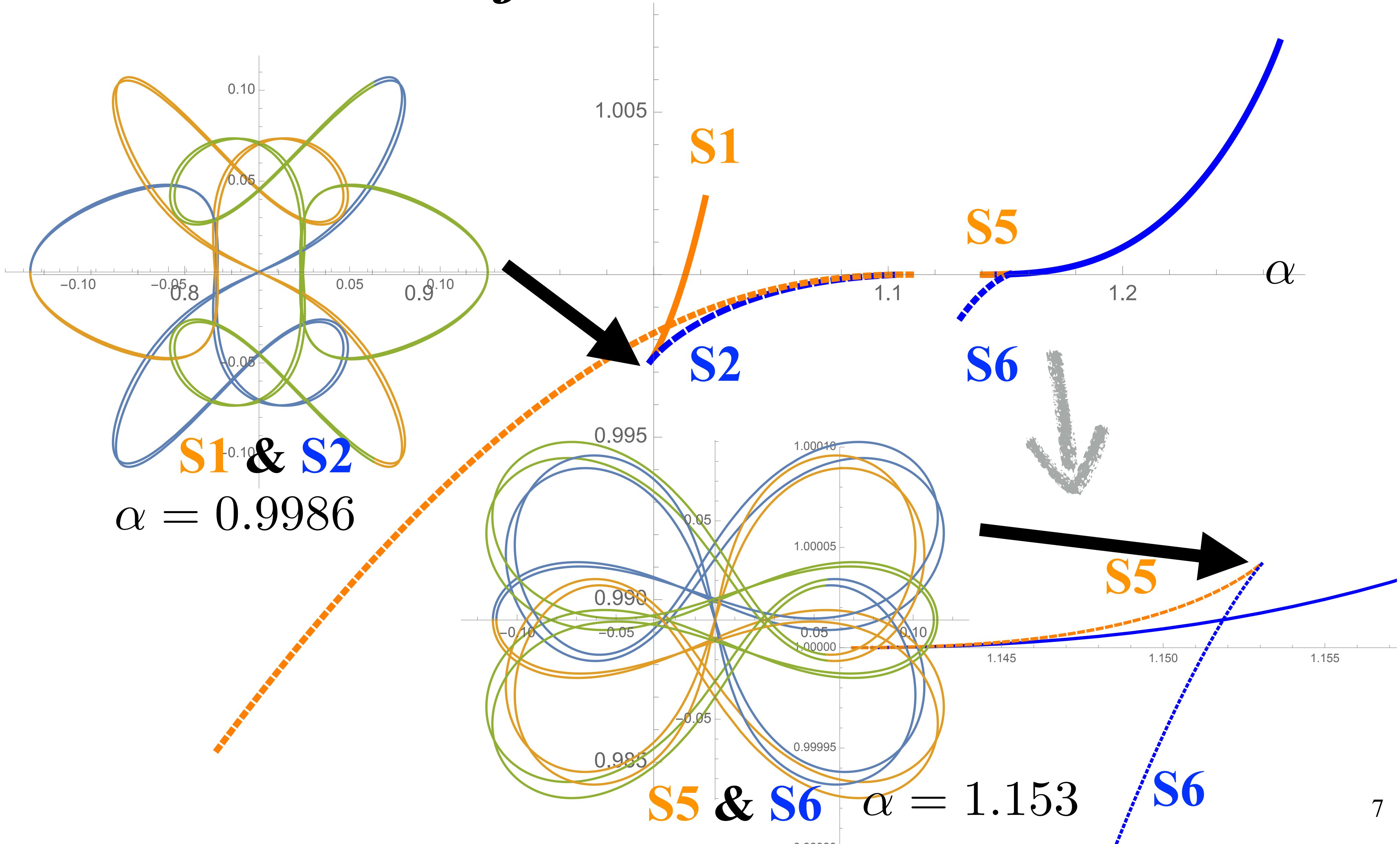
we traced the solutions



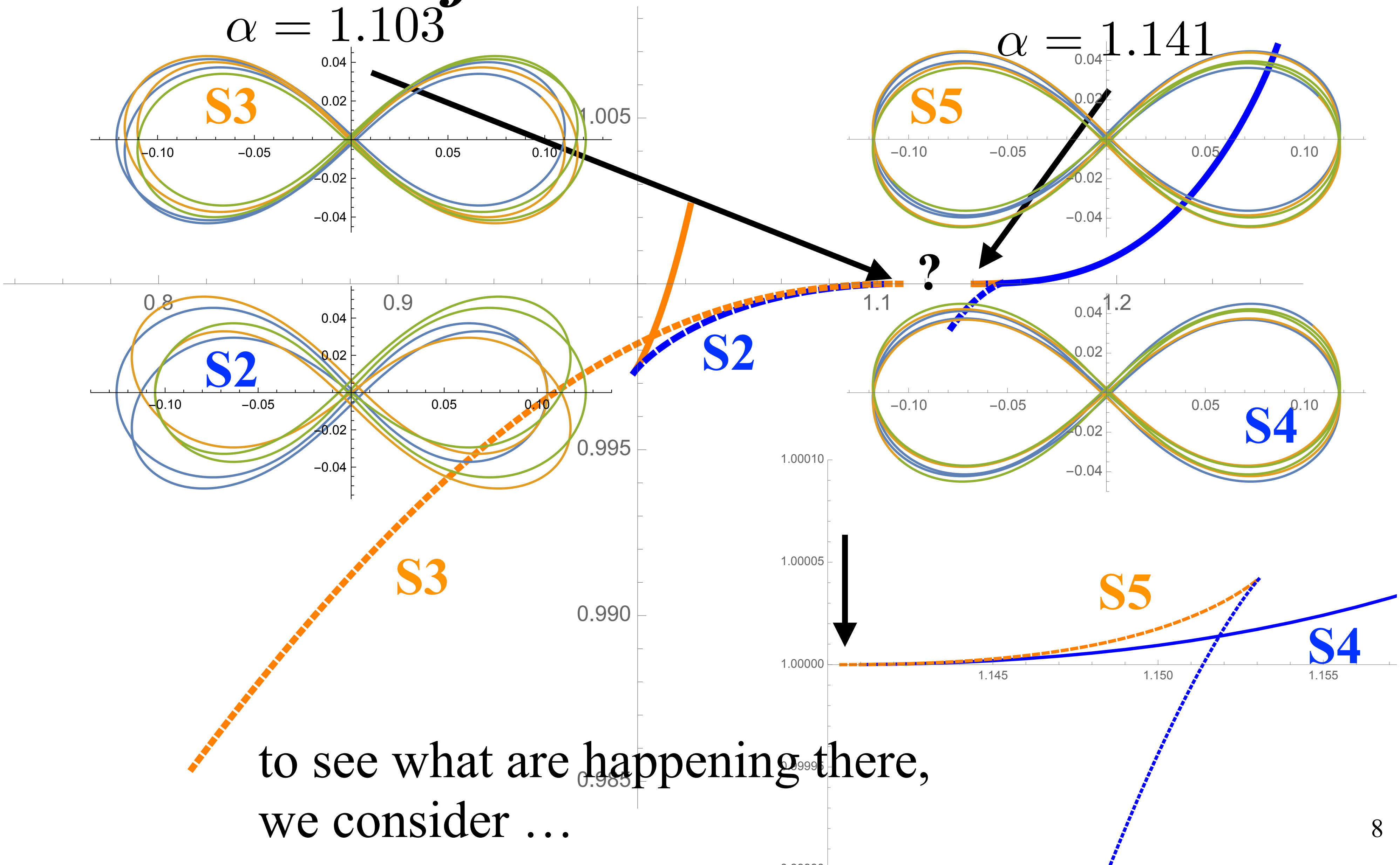
Dead ends are collision



Pair creation/annihilation of solutions



Pair creation/annihilation of solutions ?



to see what are happening there,
we consider …

Hessian, the second derivative of Action

$$S[q+\delta q] = S[q] + \frac{1}{2}\int_0^T dt\ \delta q H \delta q + \dots$$

at a solution $\delta S = 0$

eigenvalue and eigenfunction $H\Psi = \lambda\Psi,\ \Psi = \begin{pmatrix}\delta q_0\\ \delta q_1\\ \delta q_2\end{pmatrix}$

in the function space, choreographic & figure-eight symmetry

namely,
- periodic in T
- choreographic
- time reversal $t \to -t$
- time shift $t \to t + T/2$

Morse index and index theorem

Morse index i_n is the number of negative eigenvalues of Hessian for solution n.

$$\sum_n (-1)^{i_n} = \chi(M)$$: Euler character

The eigenvalues λ_m depend on α. So, i_n do.

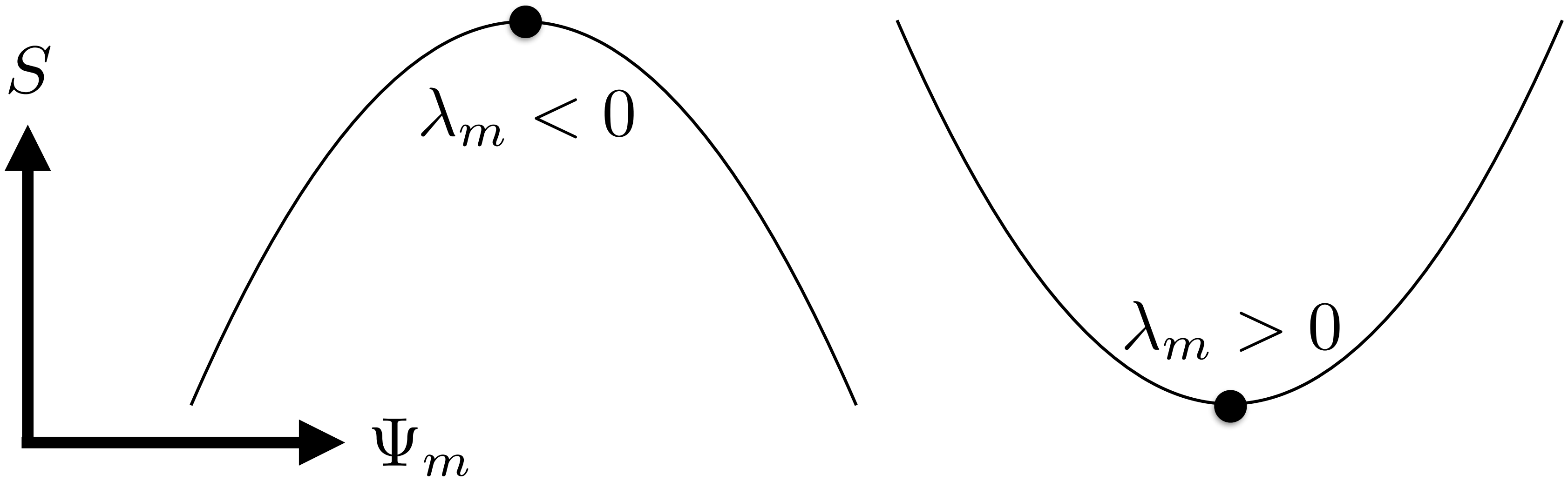
And, solution can be created or annihilated.

However, the sum must be constant.

Action and eigenvalue

$$H\Psi_m = \lambda_m \Psi_m$$

$$S[q + x\Psi_m] = S[q] + \frac{\lambda_m}{2}x^2 + ax^3 + \ldots$$



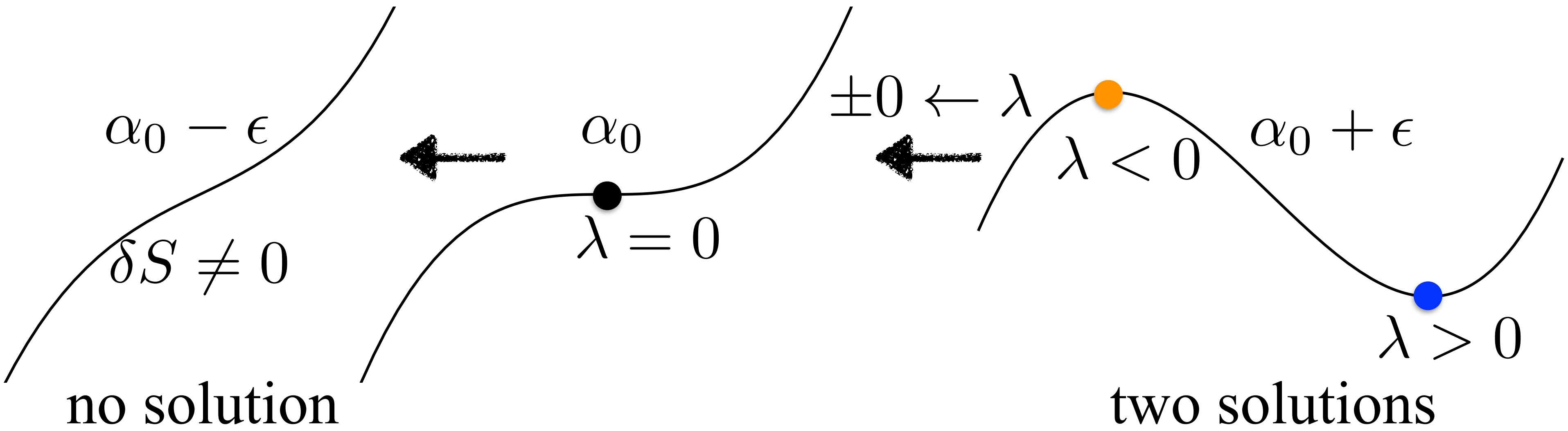
almost all eigenvalues are positive
only a few are zero or negative

if an eigenvalue vanish, then …

expected behaviour of Action "before" and "after" a zero eigenvalue

if an eigenvalue $\lambda_m = 0,$

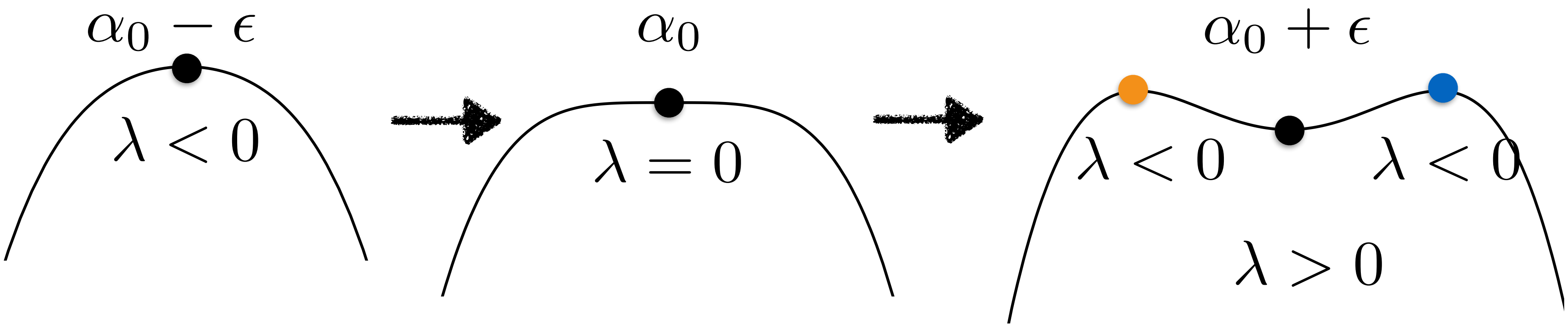
$$S[q + x\Psi_m] = S[q] + ax^3 + \dots \quad \text{at an} \quad \alpha_0$$



$$0 = (-1)^1 + (-1)^0$$

expected behaviour of Action "before" and "after" a zero eigenvalue

if $S[q + x\Psi_m] = S[q] + bx^4 + \ldots$ at an α_0



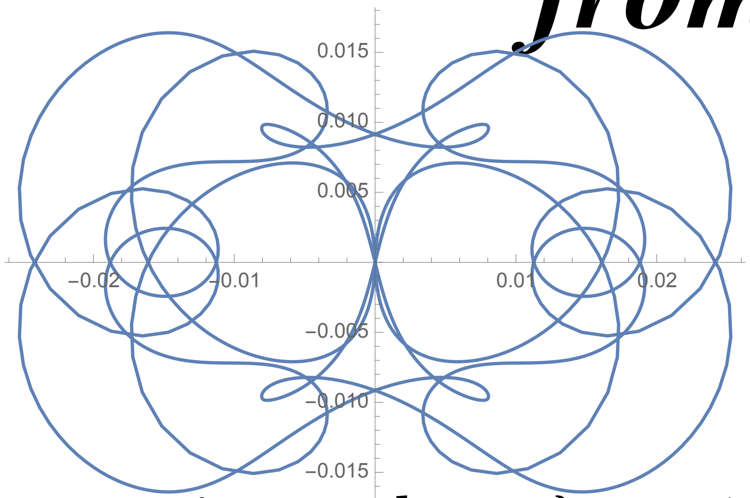
one solution

three solutions

pair creation from one solution

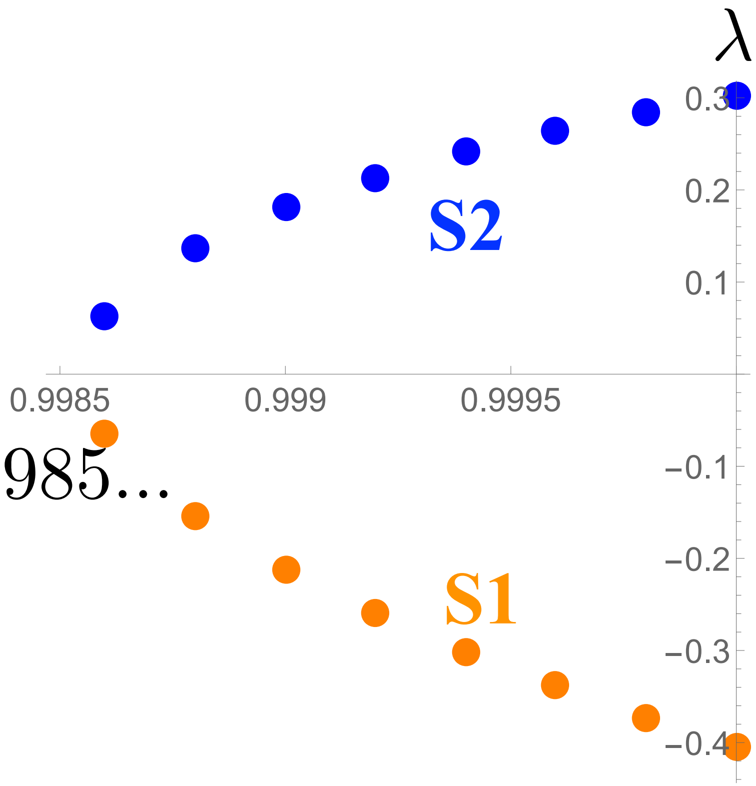
$$(-1)^1 = (-1)^0 + (-1)^1 + (-1)^1$$

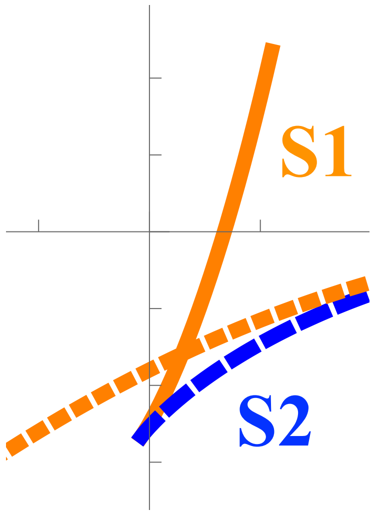
pair creation/annihilation from/to no solution

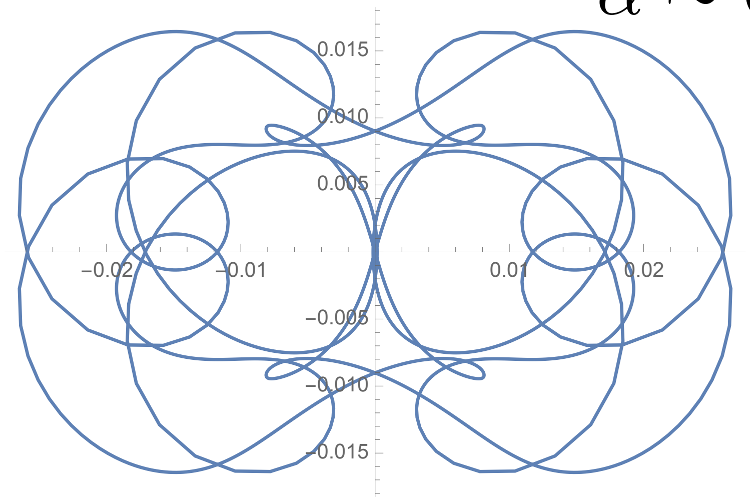


eigenvalues $\lambda \to \pm 0$

$\alpha \sim 0.9985...$



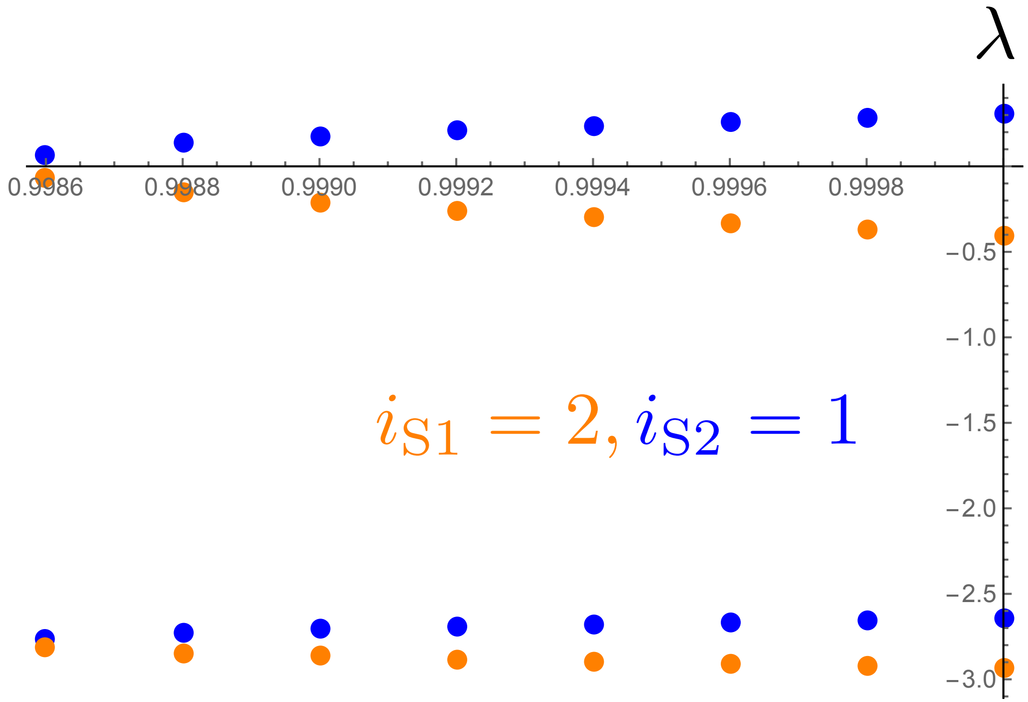




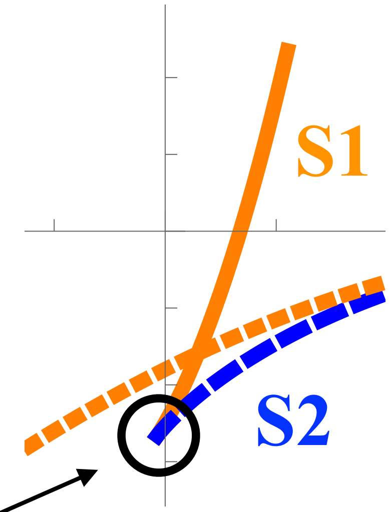
eigenfunctions and eigenvalues

→ see page 12

pair creation from no solution



$$i_{\mathrm{S1}} = 2, i_{\mathrm{S2}} = 1$$

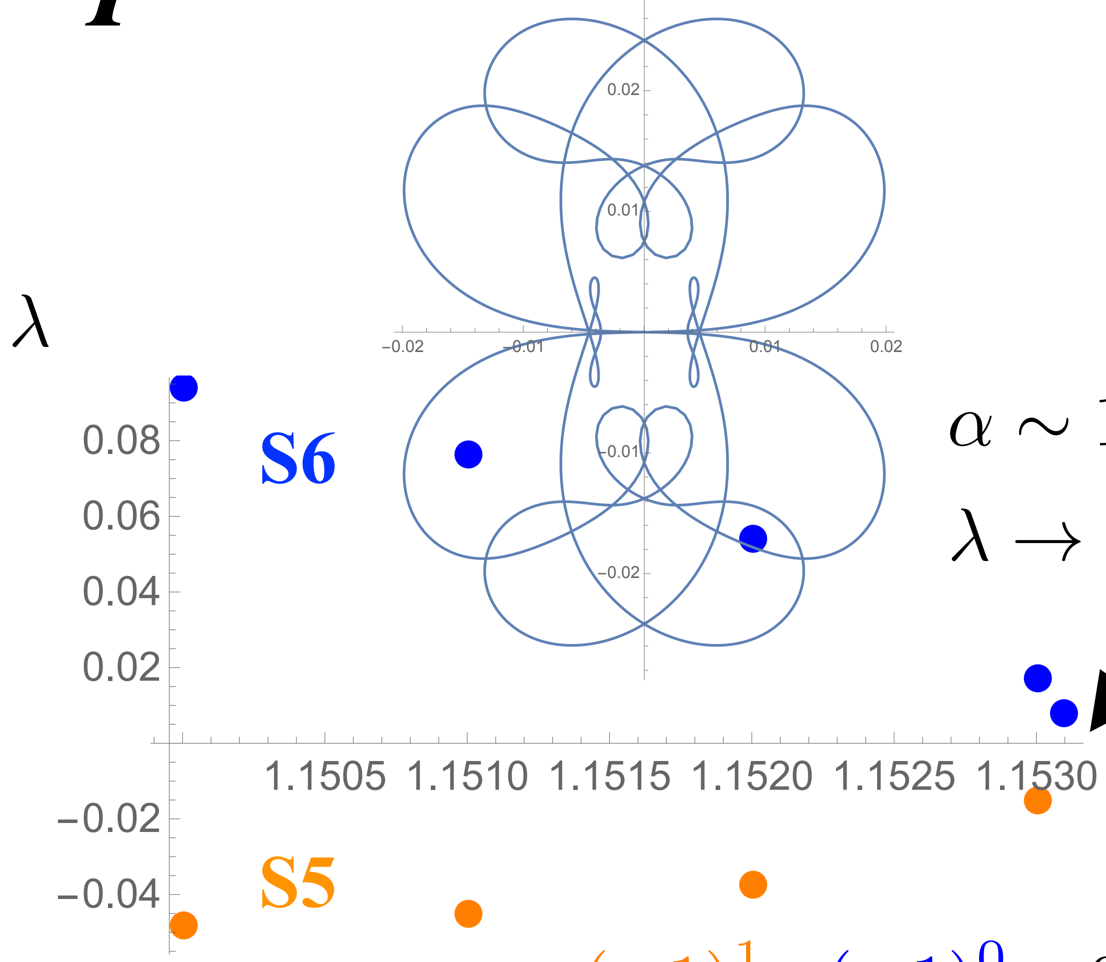


$$0 = (-1)^2 + (-1)^1$$

no solution ⟷ **S1** and **S2**

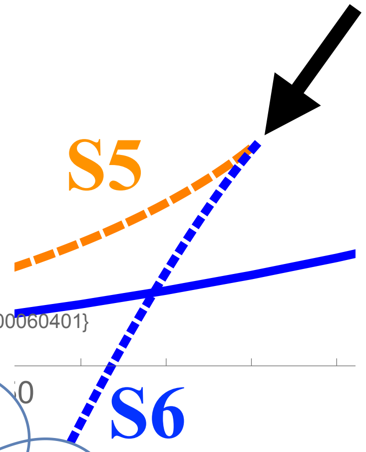
➡ see page 12

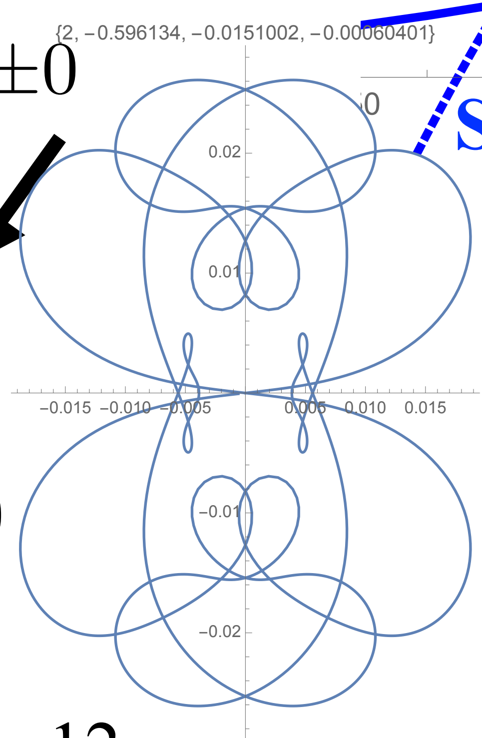
pair annihilation to no solution



$\alpha \sim 1.153...$

$\lambda \to \pm 0$

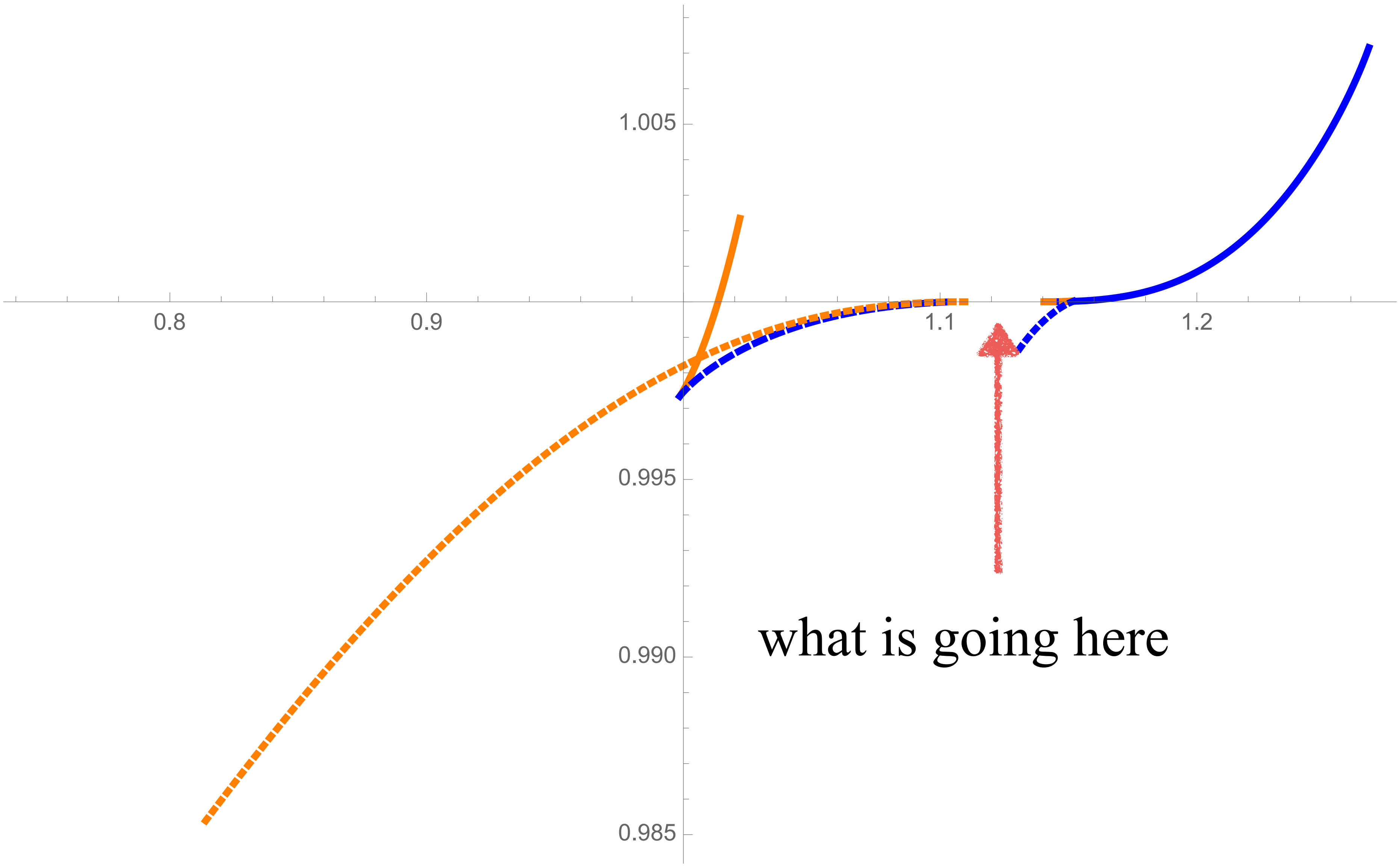




$(-1)^1 + (-1)^0 = 0$

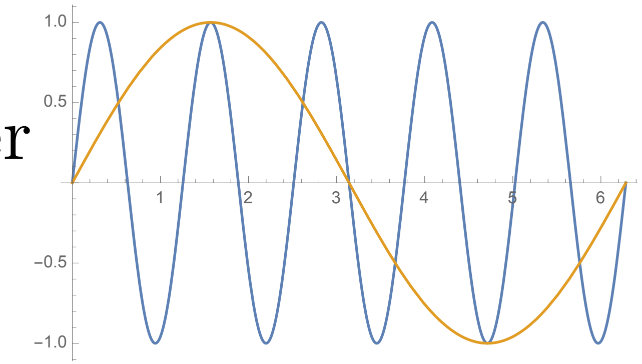
eigenfunctions and eigenvalues

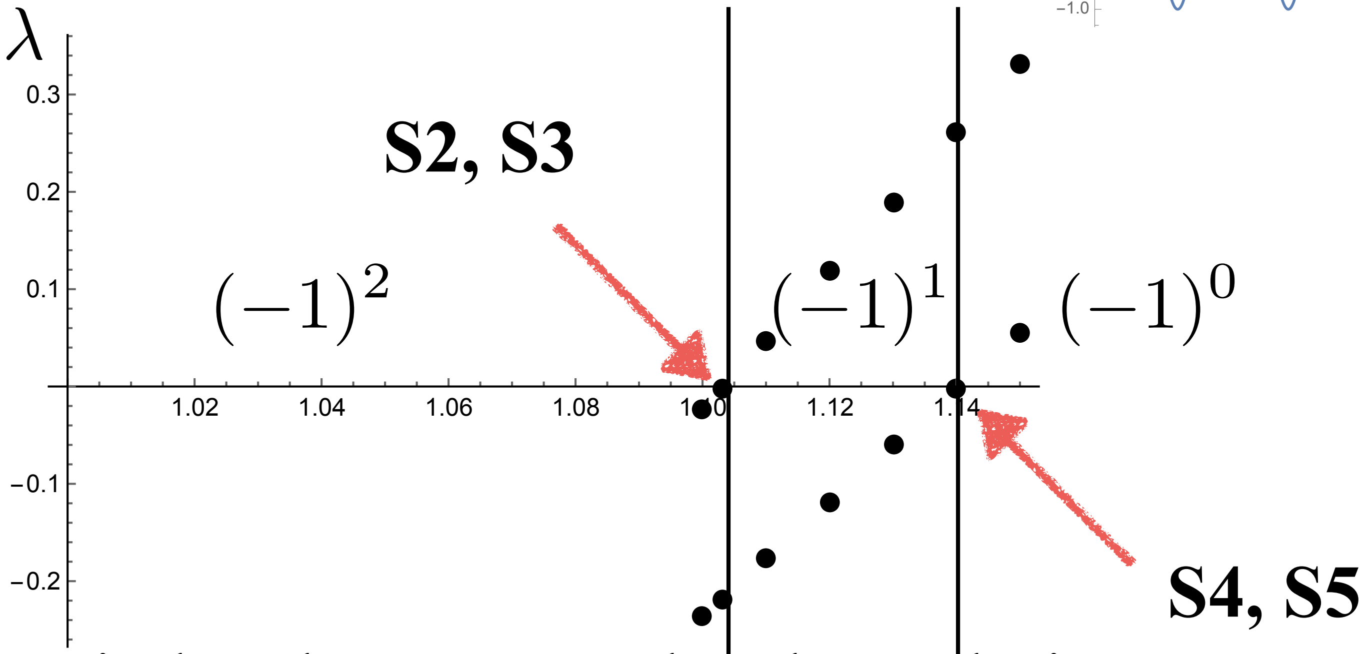
see page 12


1.005
0.995
0.990
0.985
0.8
0.9
1.1
1.2
what is going here

pair annihilation/creation to/from figure-eight(5)

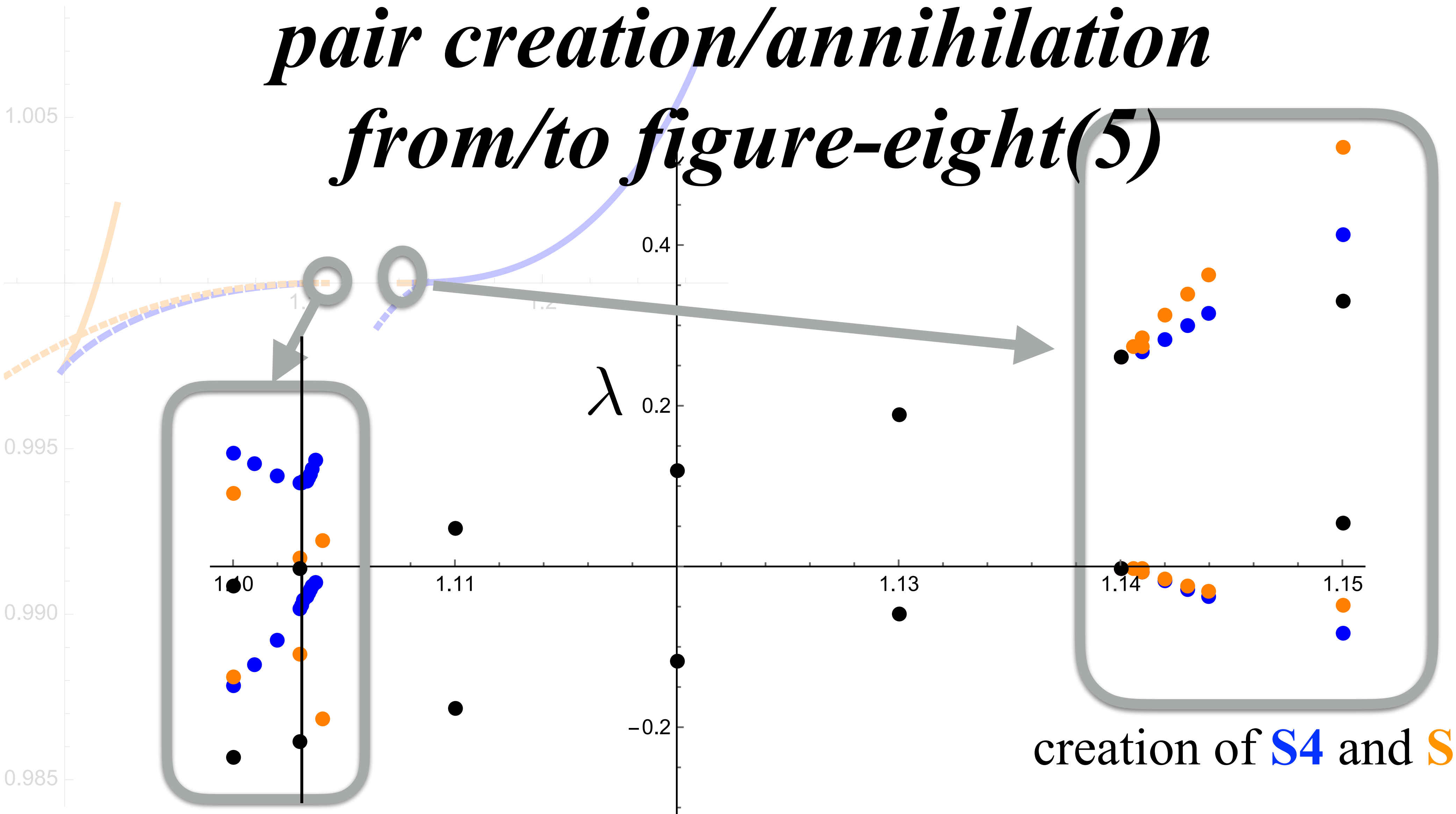
figure-eight(5) is NOT an action minimiser in function space $T = 5T_{\text{fig8}(5)}$





index theorem needs other solutions

pair creation/annihilation from/to figure-eight(5)

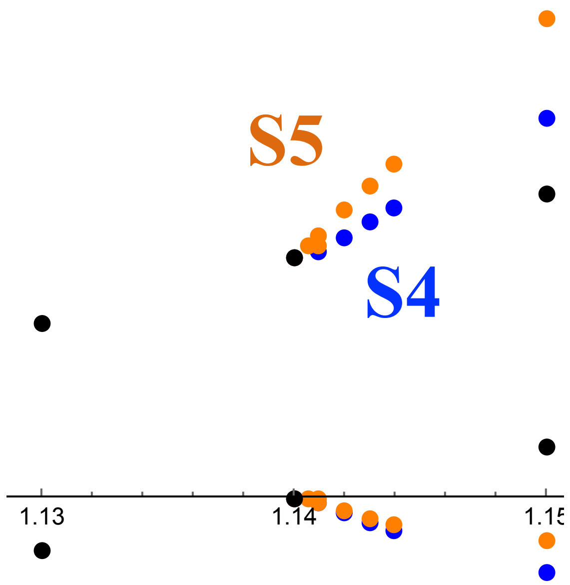


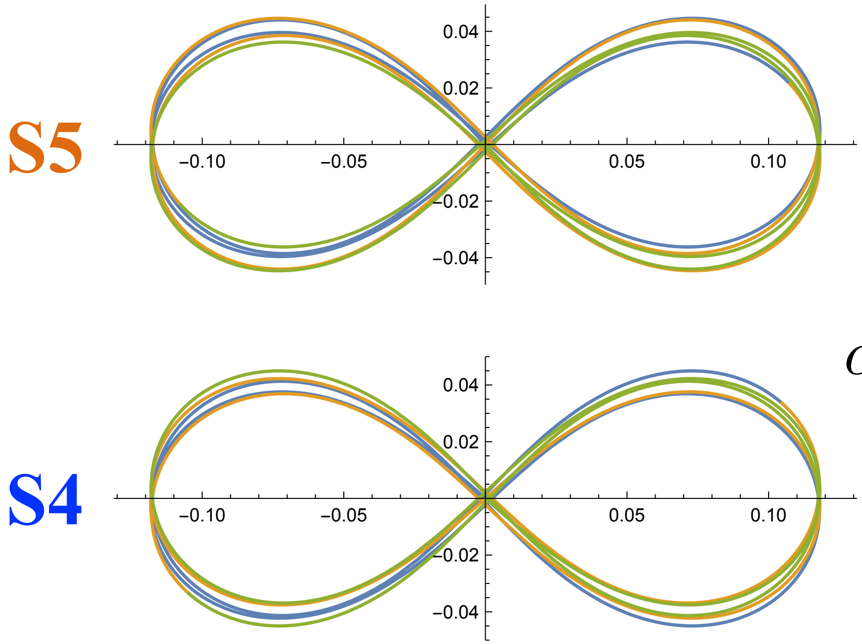
creation of **S4** and **S5**

annihilation of **S2** and **S3** ???

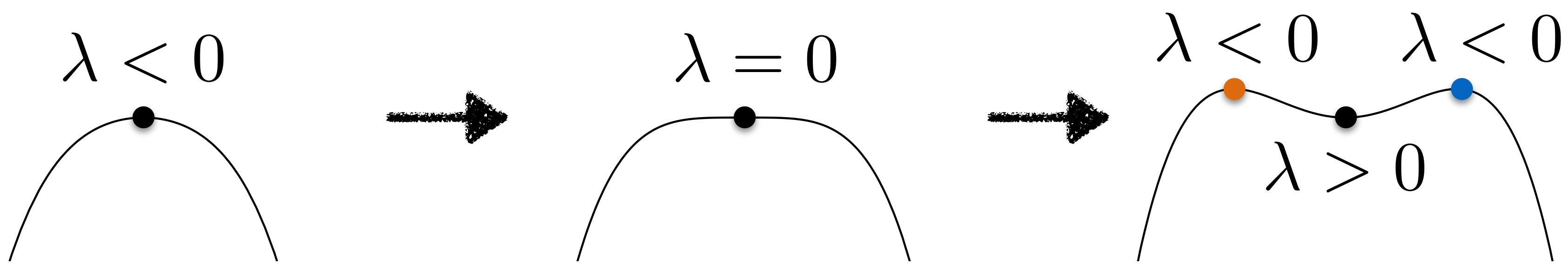
pair creation from figure-eight(5)

is clear for $\alpha \lesssim 1.141$





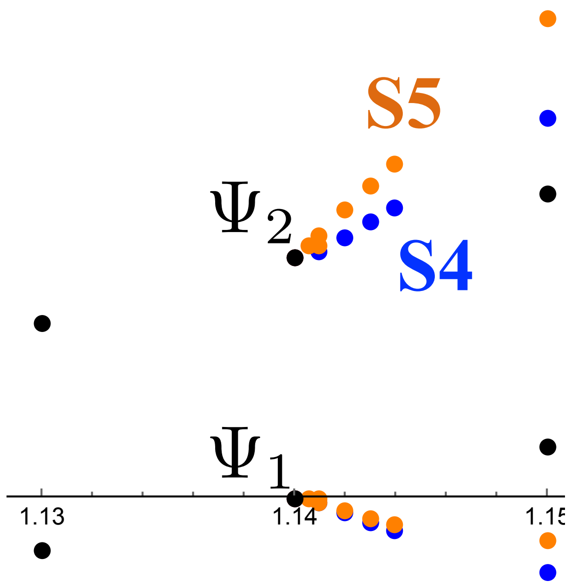
$\alpha = 1.141$



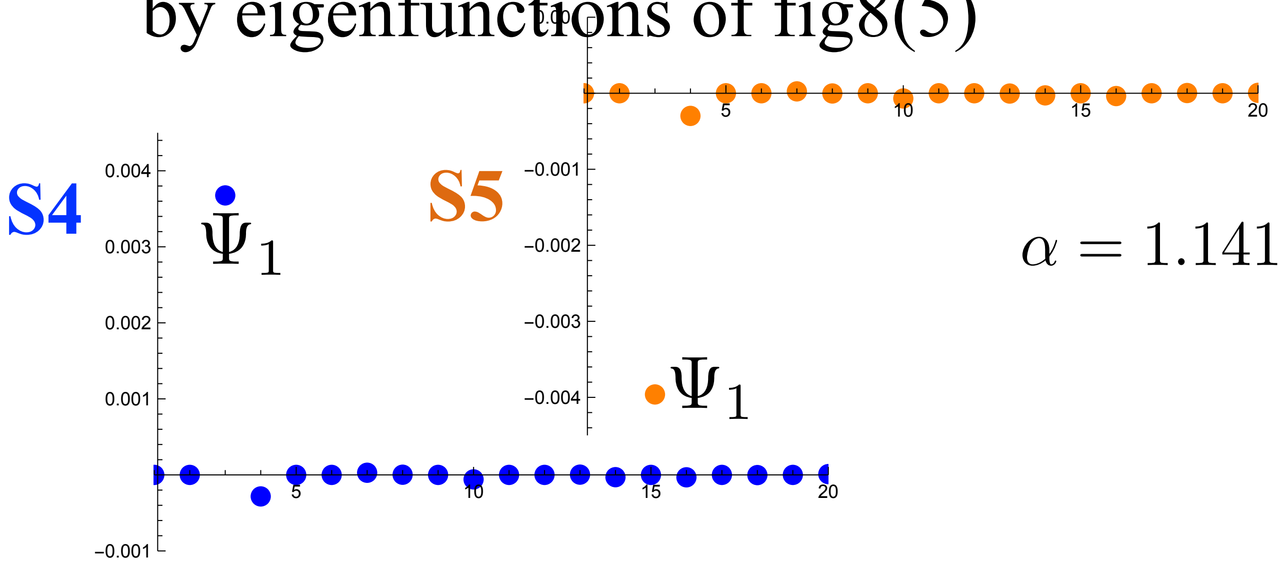
see page 13

pair creation from figure-eight(5)

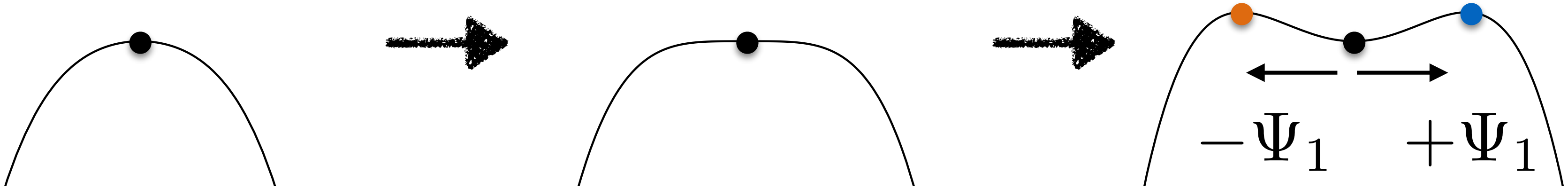
is clear for $\alpha \sim 1.141$



expansion $q_{\mathrm{Sn}} - q_{\mathrm{fig8(5)}}$
by eigenfunctions of fig8(5)



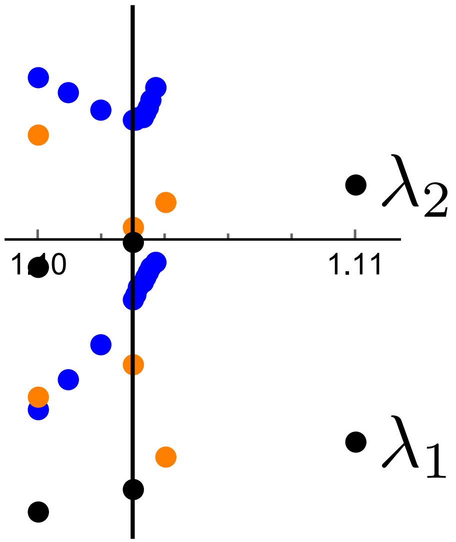
therefore, the problem is almost ONE dimensional

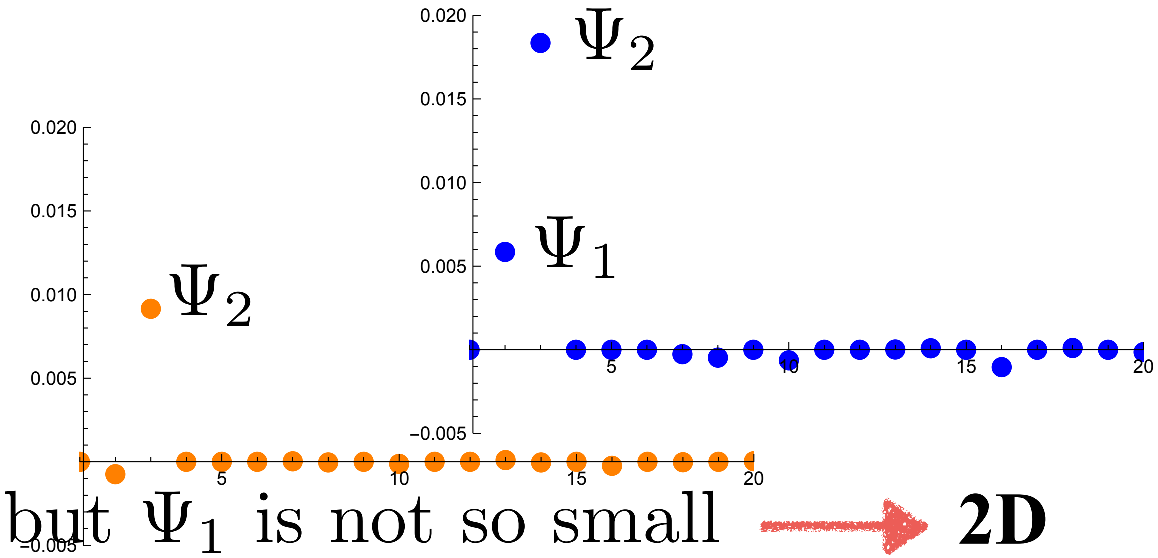


pair annihilation to figure8(5)

is NOT clear for $\alpha \gtrsim 1.03$

expansion $q_{\mathrm{Sn}} - q_{\mathrm{fig8(5)}}$
by eigenfunctions of fig8(5)





the main term is Ψ_2, but Ψ_1 is not so small → **2D**

same side of Ψ_2

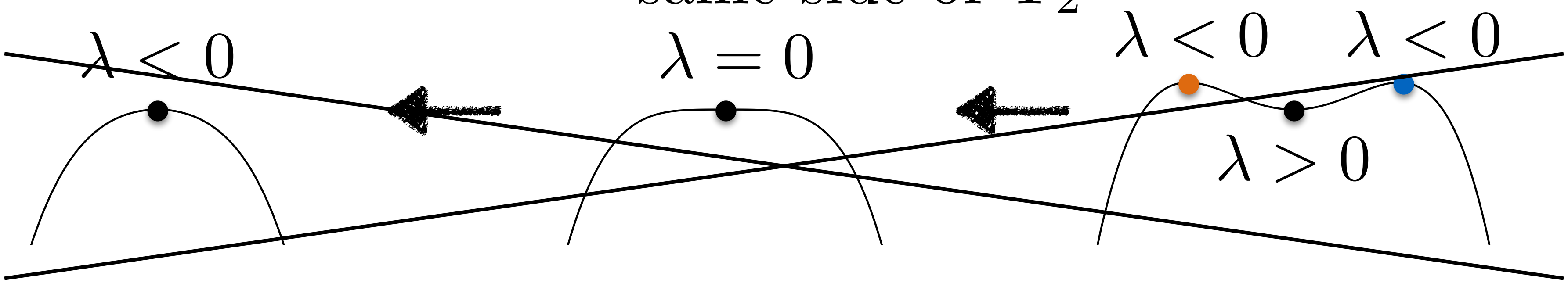
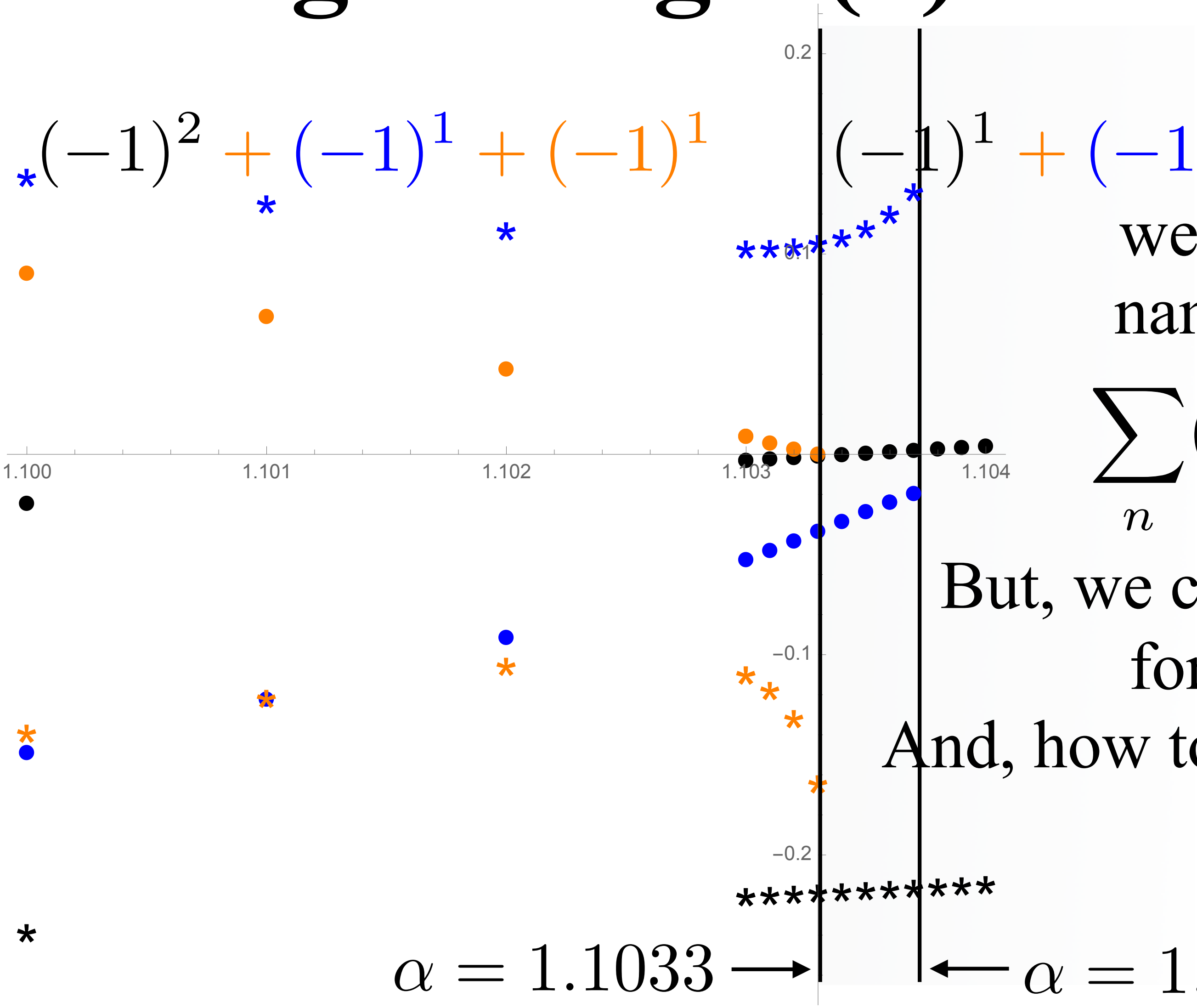


figure-eight(5) *and* S2, S3



$(-1)^2 + (-1)^1 + (-1)^1$ $\quad$ $(-1)^1 + (-1)^1 + (-1)^2$

we need this **term**, namely **S3**, to keep

$$\sum_n (-1)^{i_n} = \text{const.}$$

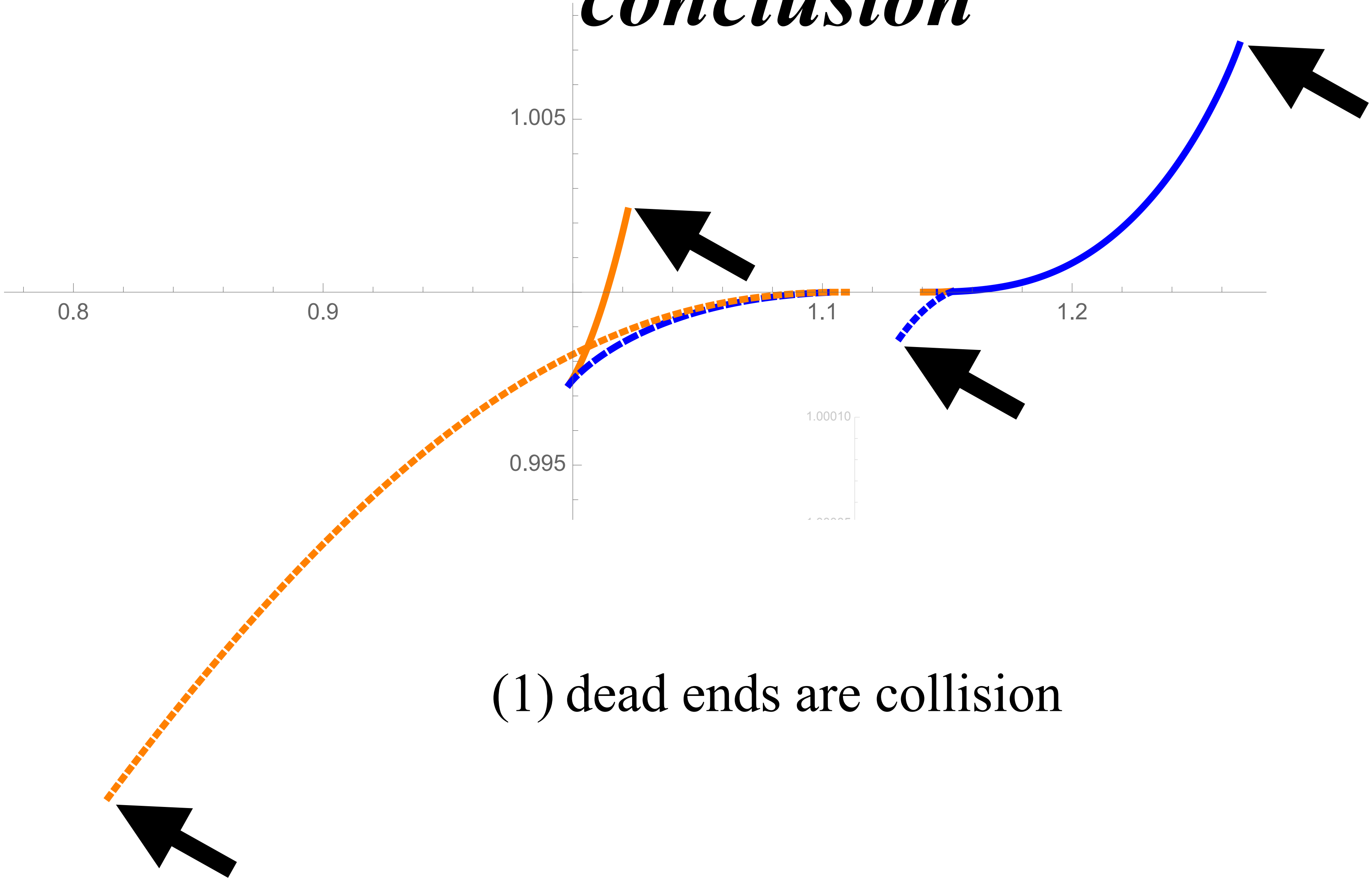
But, we couldn't trace **S3** for lager α .

And, how to annihilate them?

$\alpha = 1.1033 \rightarrow$ $\quad$ $\leftarrow \alpha = 1.1037$

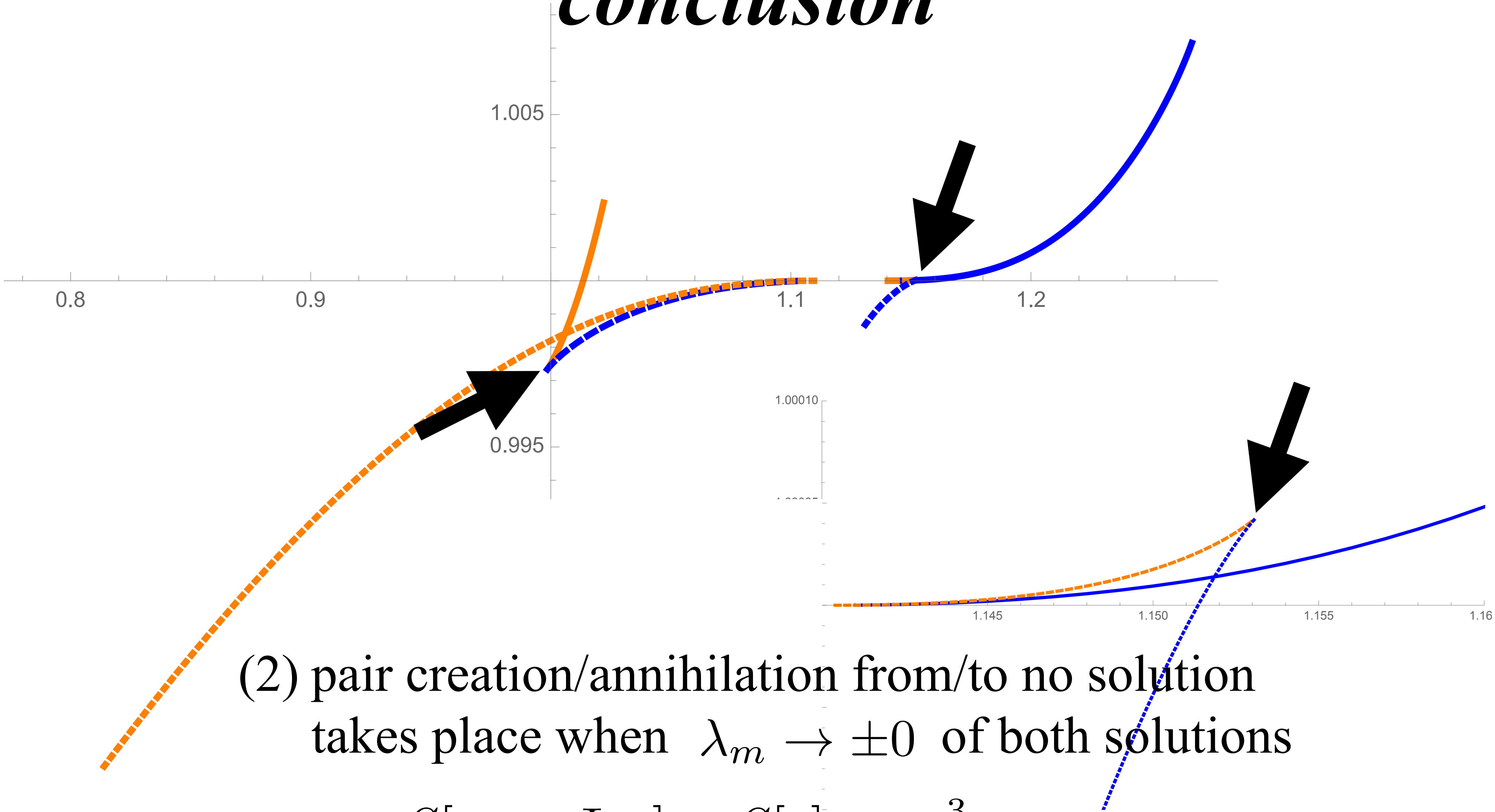
$\Delta\alpha \sim 4 \times 10^{-4}$

figure-eight(5) and slalom solutions conclusion



(1) dead ends are collision

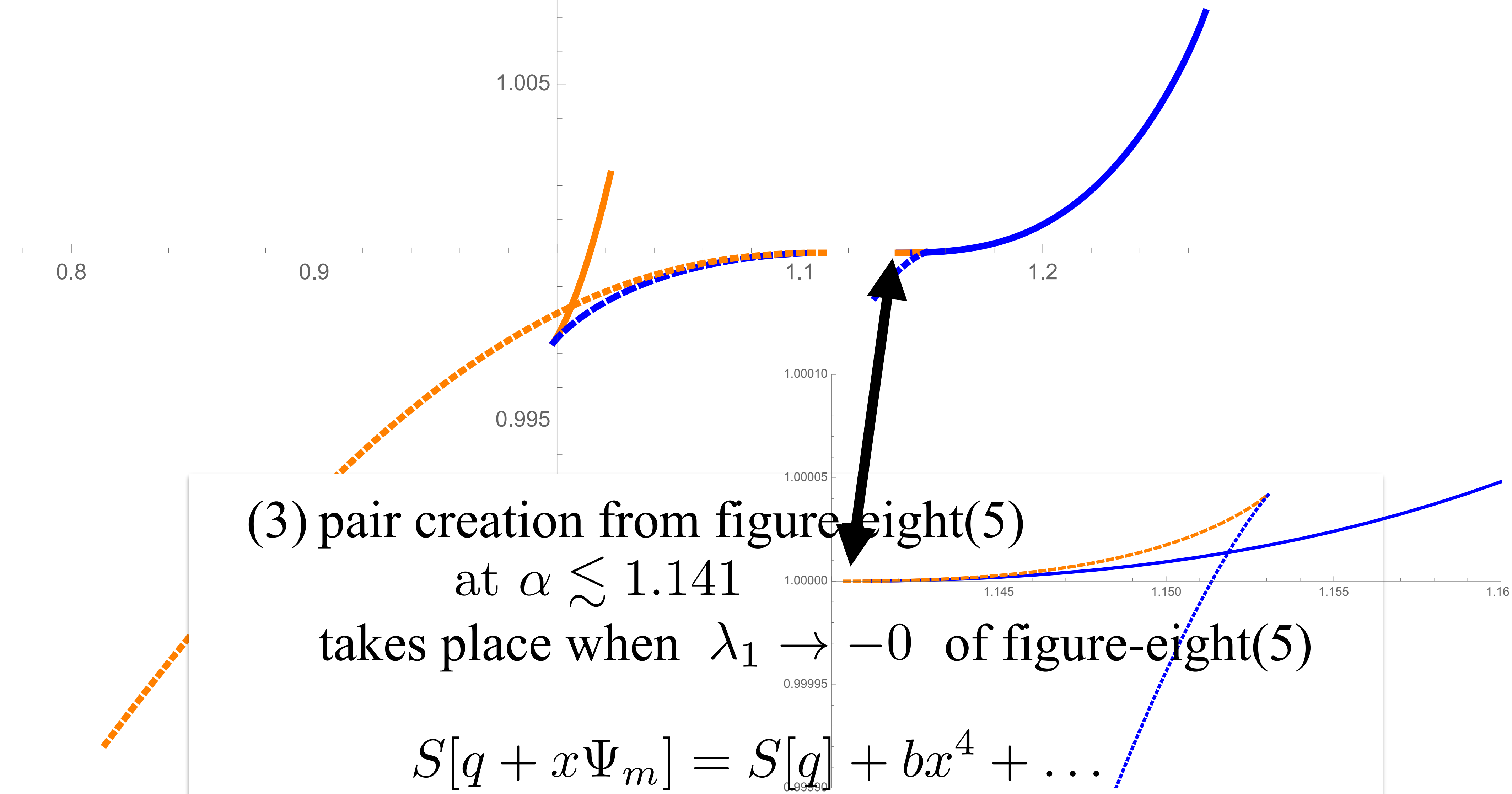
figure-eight(5) and slalom solutions conclusion



(2) pair creation/annihilation from/to no solution takes place when $\lambda_m \to \pm 0$ of both solutions

$$S[q + x\Psi_m] = S[q] + ax^3 + \dots$$

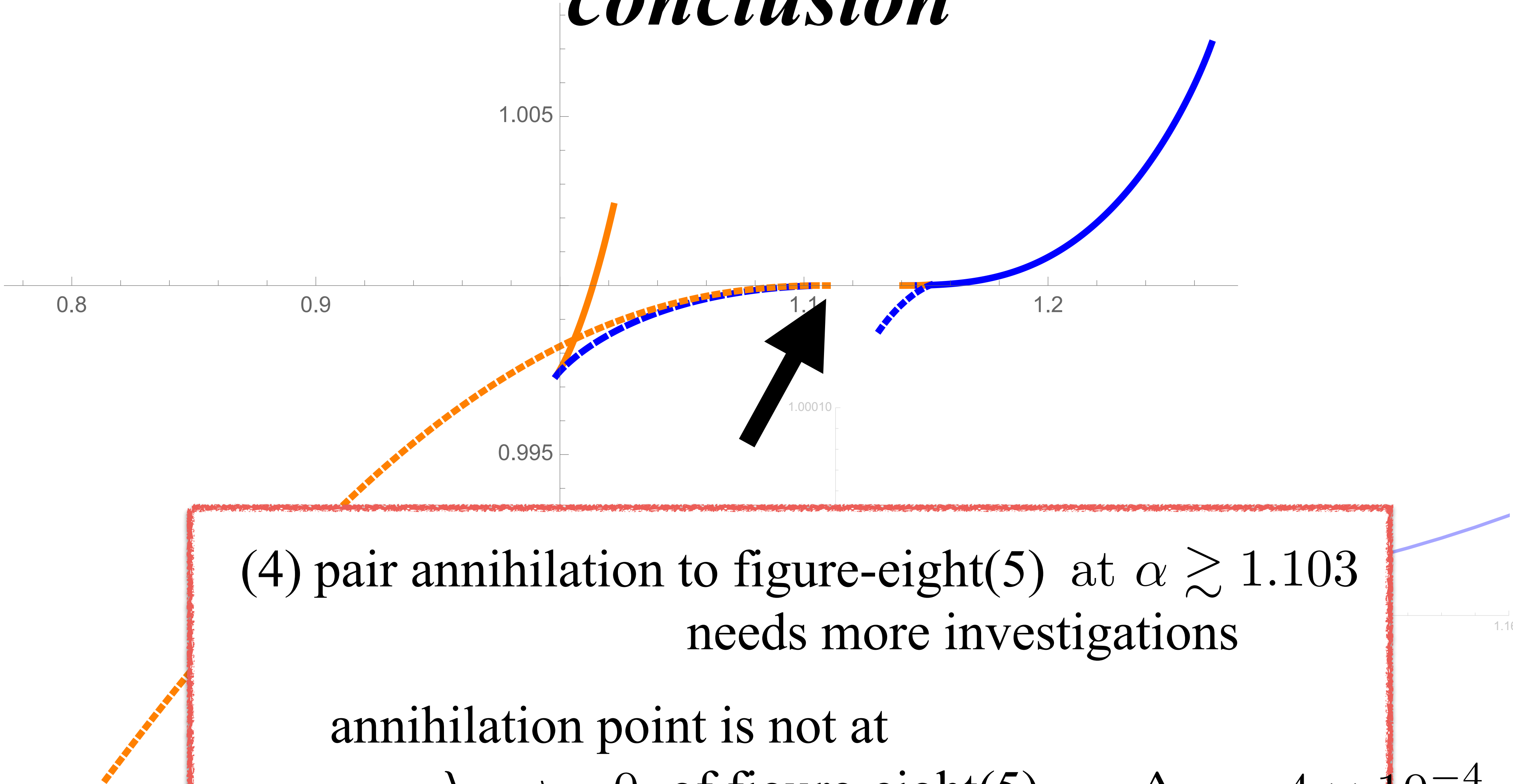
figure-eight(5) and slalom solutions conclusion



(3) pair creation from figure-eight(5)
at $\alpha \lesssim 1.141$
takes place when $\lambda_1 \to -0$ of figure-eight(5)

$$S[q + x\Psi_m] = S[q] + bx^4 + \dots$$

figure-eight(5) and slalom solutions conclusion



(4) pair annihilation to figure-eight(5) at $\alpha \gtrsim 1.103$
needs more investigations

annihilation point is not at
$\lambda_2 \to -0$ of figure-eight(5) $\Delta\alpha \sim 4 \times 10^{-4}$

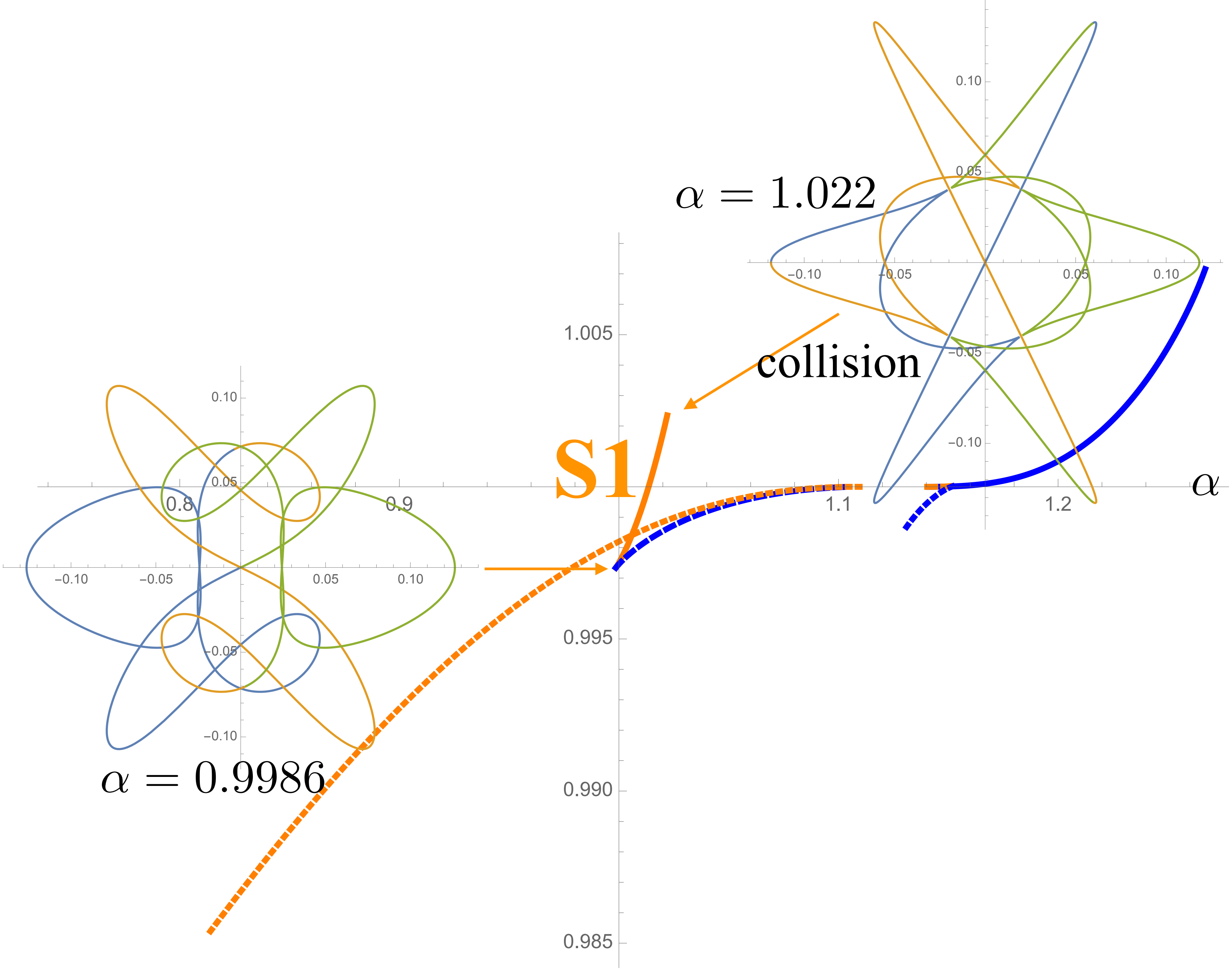
figure-eight(5) and slalom solutions conclusion & question

1. Dead ends are collision
2. Pair creation/annihilation from/to no solution takes place when $\lambda_m \to \pm 0$ of both solutions
3. Pair creation from figure-eight(5) at $\alpha \sim 1.141$ takes place when $\lambda_m \to -0$ of figure-eight(5) solution
4. Pair annihilation to figure-eight(5) at $\alpha \sim 1.103$ needs more investigations
   Annihilation point is not at $\lambda_m = 0$ of figure-eight(5)

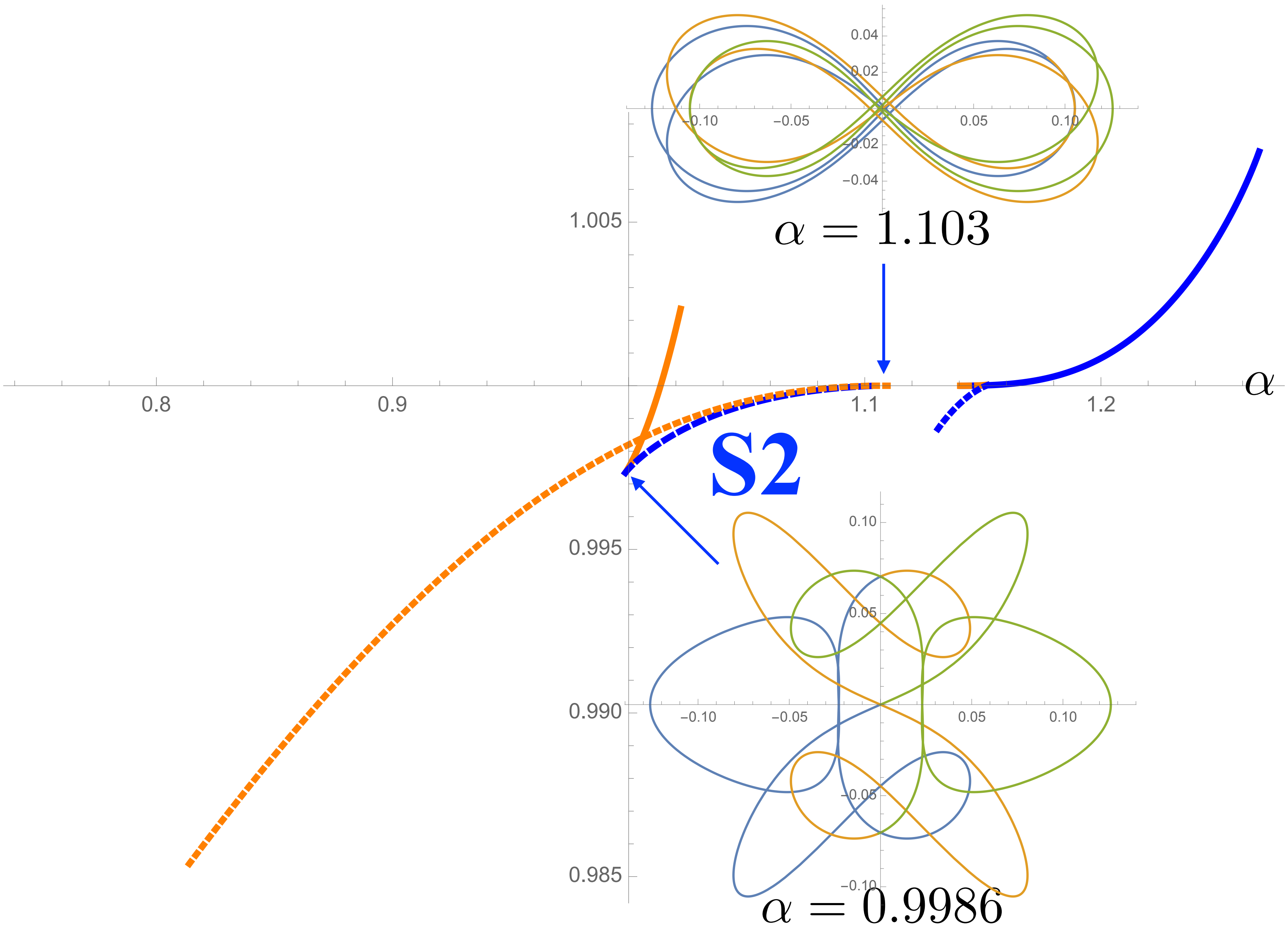
Q: what is happing there? How to annihilate them?

We need some theoretical approaches.

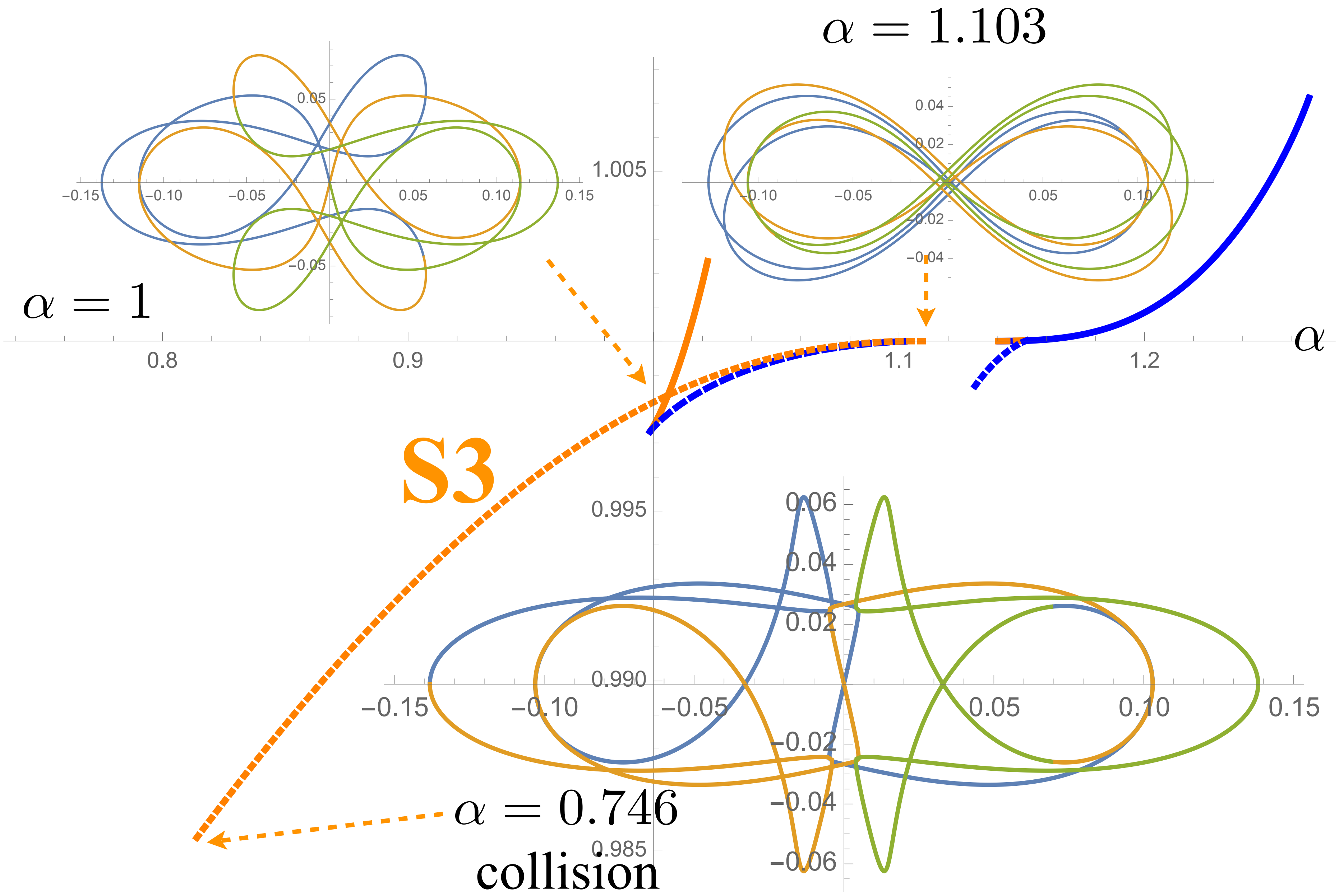
appendix
S1~S6

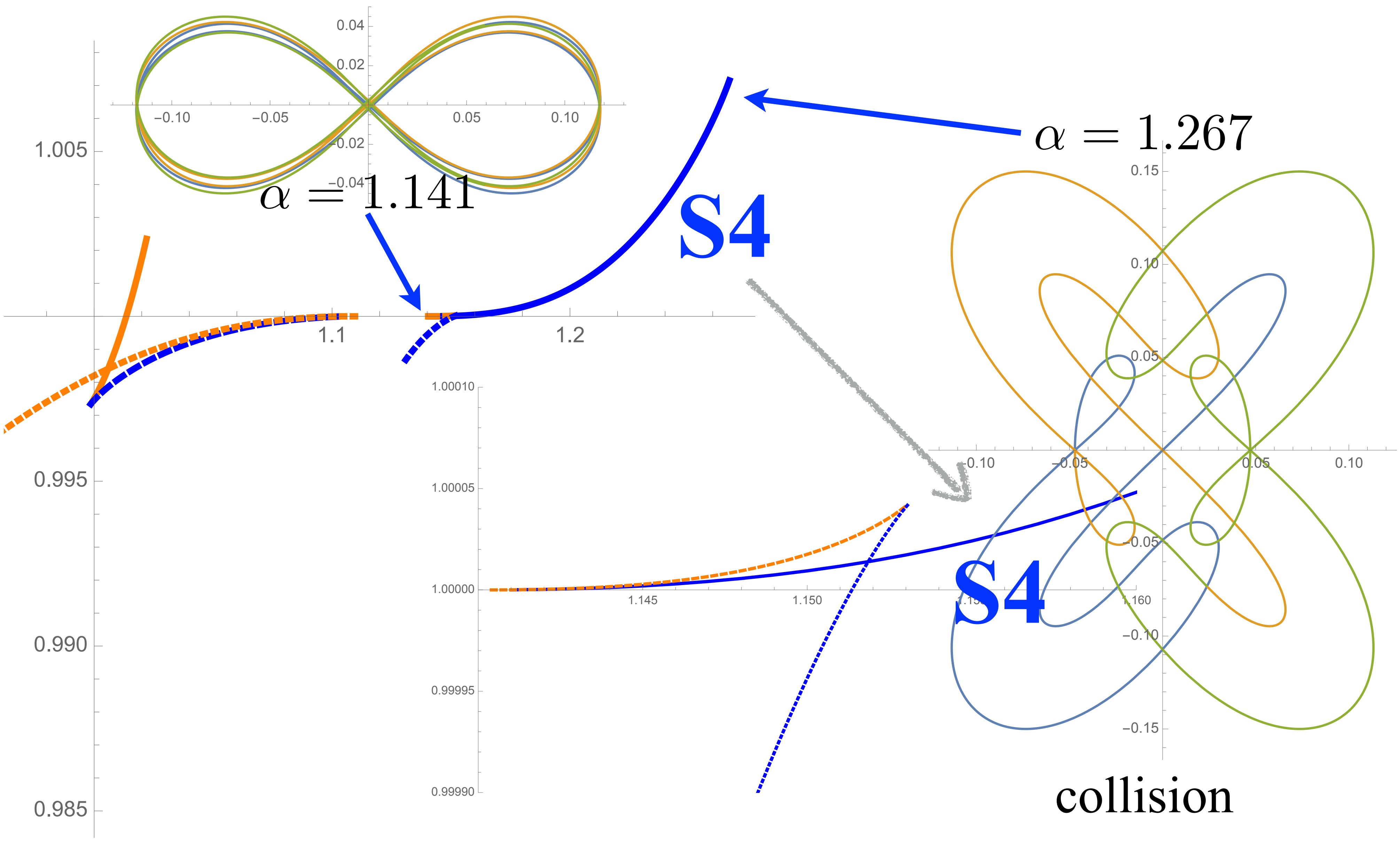


α = 1.022
collision
S1
α = 0.9986
α
1.005
0.995
0.990
0.985
0.8
0.9
1.1
1.2
0.10
0.05
−0.05
−0.10

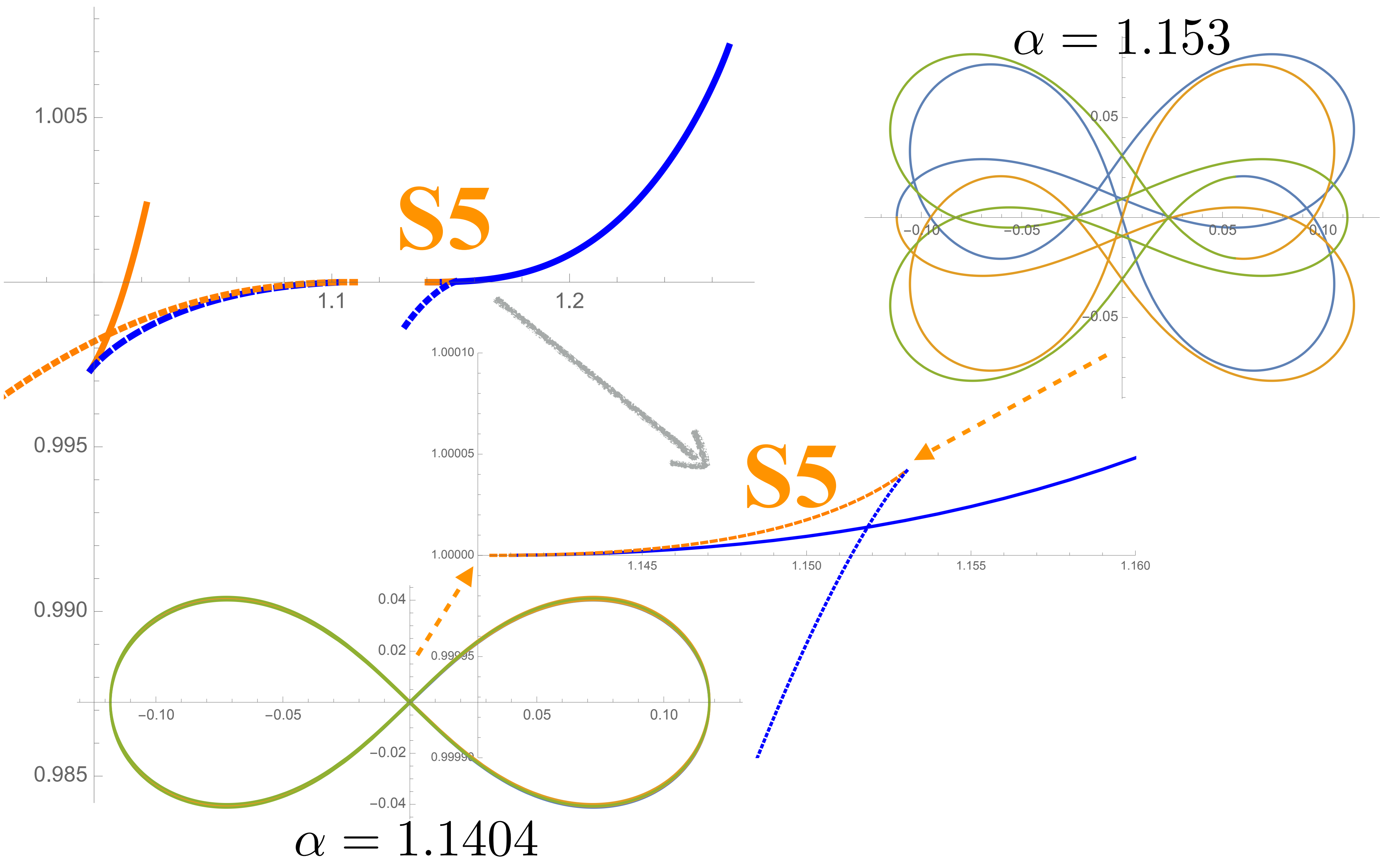


0.04
0.02
−0.10
−0.05
0.05
0.10
−0.02
−0.04
1.005
$\alpha = 1.103$
0.8
0.9
1.1
1.2
α
S2
0.995
0.10
0.05
0.990
−0.10
−0.05
0.05
0.10
−0.05
0.985
−0.10
$\alpha = 0.9986$

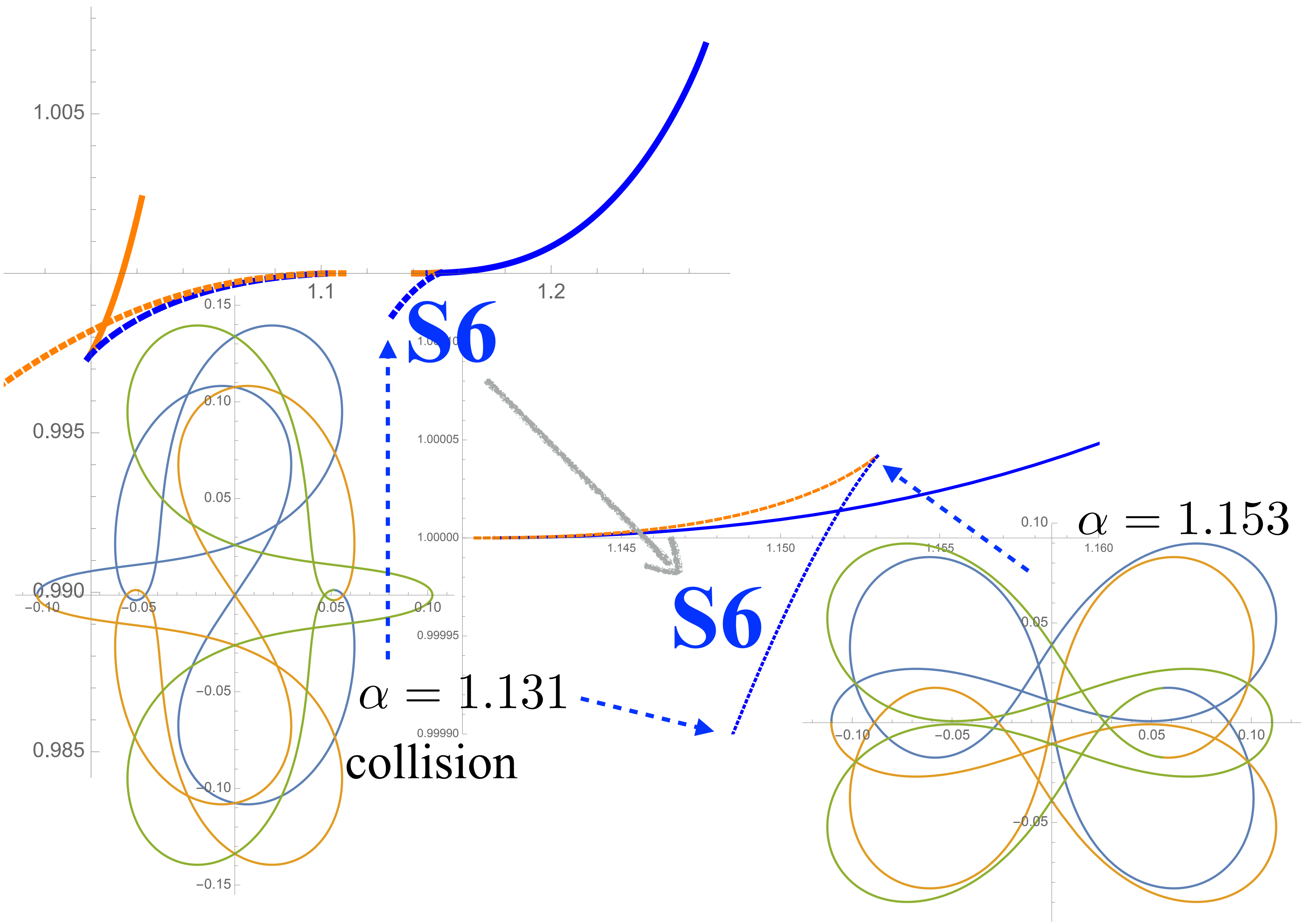

$\alpha = 1.103$
$\alpha = 1$
S3
α
$\alpha = 0.746$
collision
1.005
0.995
0.990
0.985
0.8
0.9
1.1
1.2



α = 1.267
α = 1.141
S4
S4
collision
1.005
0.995
0.990
0.985
1.1
1.2
1.00010
1.00005
1.00000
0.99995
0.99990
1.145
1.150
1.160
0.04
0.02
−0.02
−0.04
−0.10
−0.05
0.05
0.10
0.15
−0.15



$\alpha = 1.153$
1.005
0.995
0.990
0.985
1.1
1.2
S5
S5
1.00010
1.00005
1.00000
1.145
1.150
1.155
1.160
0.05
−0.05
−0.10
0.10
0.04
0.02
−0.02
−0.04
$\alpha = 1.1404$



S6
S6
$\alpha = 1.131$
collision
$\alpha = 1.153$