Eigenvalues and eigenfunctions for second derivative of the action at the figure-eight and slalom solutions

Toshiaki Fujiwara, Hiroshi Fukuda, and Hiroshi Ozaki

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天体力学Ｎ体力学研究会

NAO, Mitaka

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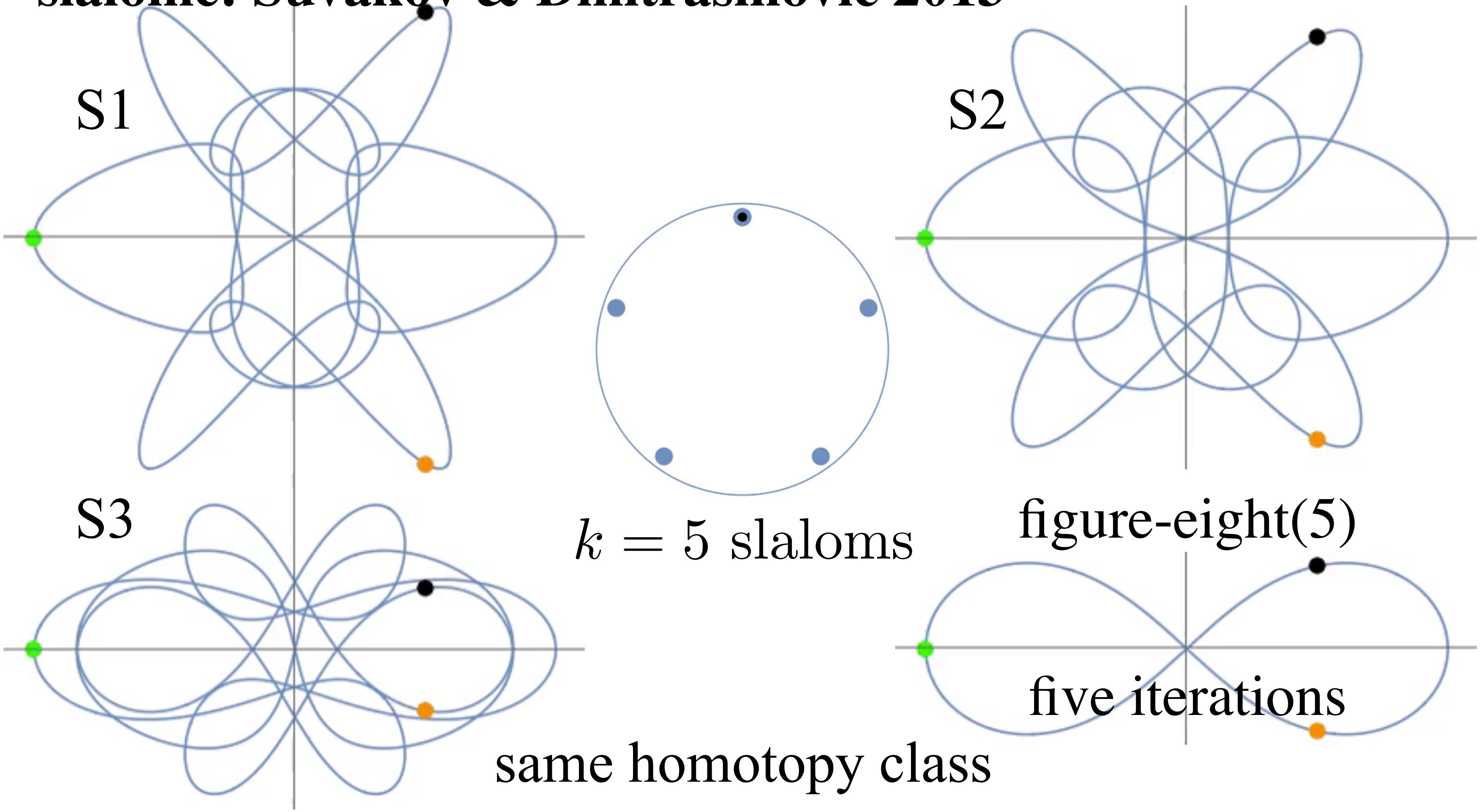
Figure-eight and slalom solutions

figure-eight: Moore 1993, Chenciner & Montgomery 2000

slalome: Šuvakov & Dmitrašinović 2013



Three-body choreography

$$L = \frac{1}{2}\sum_k \left|\frac{dq_k}{dt}\right|^2 + \frac{1}{\alpha}\sum_{i,j}\frac{1}{|q_i - q_j|^\alpha}$$

$\alpha = 1$: Newton potential

$$q_0(t) = q(t), q_1(t) = q(t + T/3), q_2(t) = q(t + 2T/3)$$

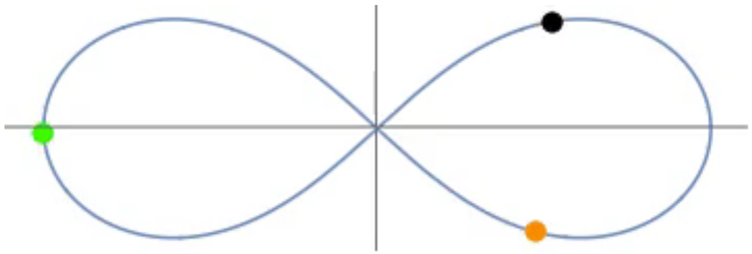
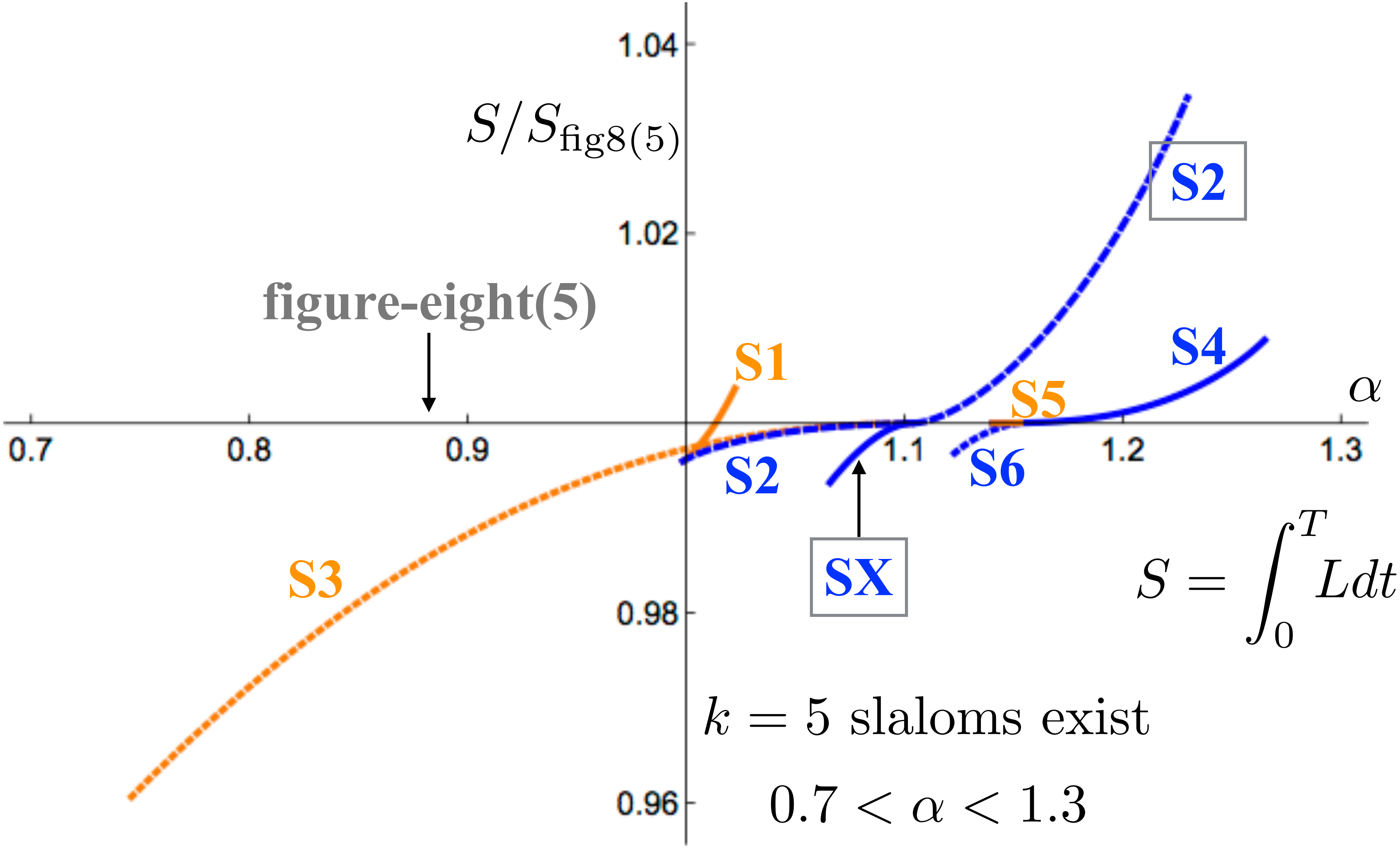


figure-eight solution
C. Moore 1993,
A. Chenciner and R. Montgomery 2000

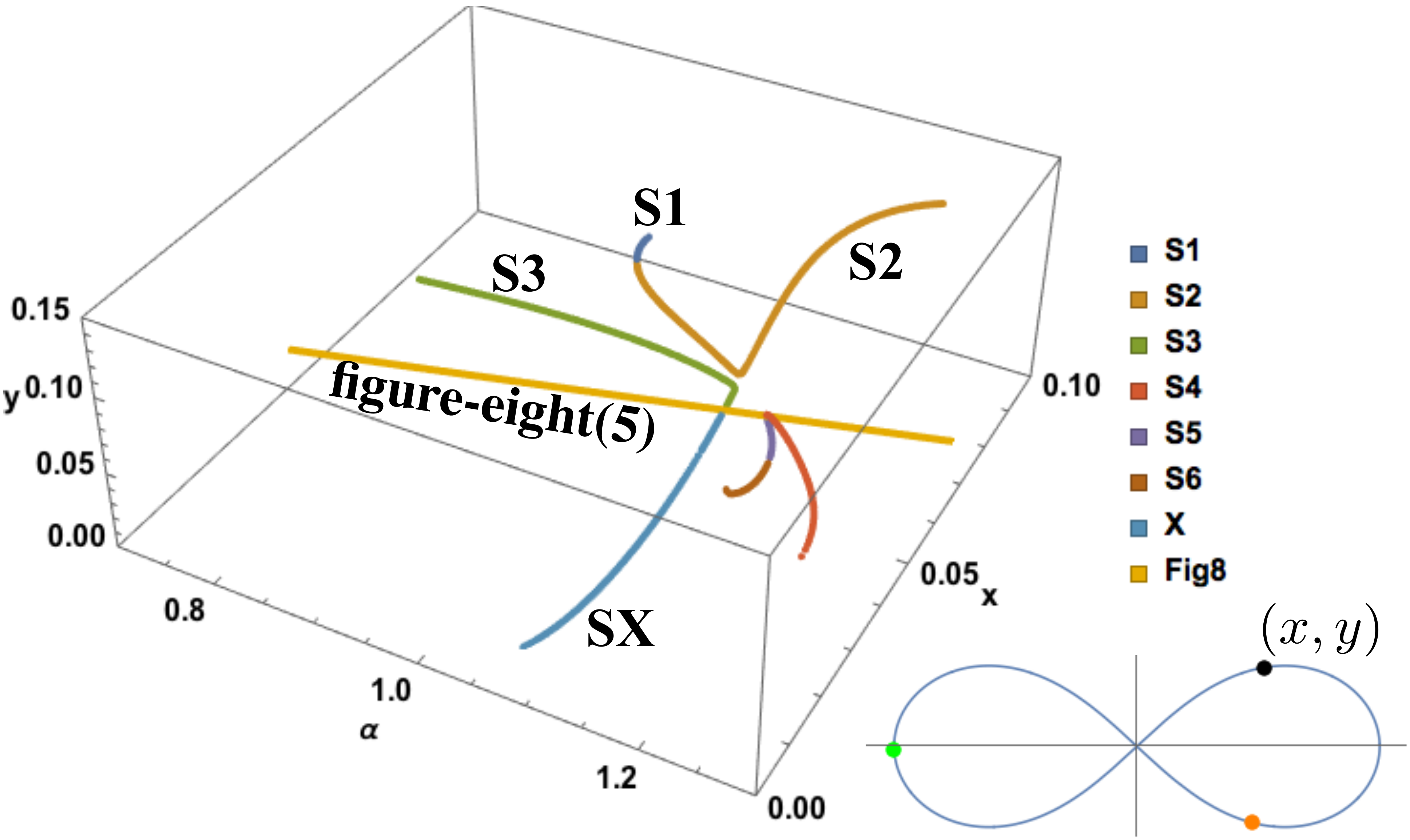
Slalom Solutions

- **Šuvakov M.**
  - **Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere, Celest. Mech. Dyn. Astron. 119, 369–377 (2014)**
- **Šuvakov M., Dmitrašinović V.**
  - **Three classes of Newtonian three-body planar periodic orbits, Phys. Rev. Lett. 110(11), 114301 (2013)**
- **Šuvakov M., Dmitrašinović V.**
  - **A guide to hunting periodic three-body orbits, Am. J. Phys. 82, 609–619 (2014)**
- **Šuvakov M., Shibayama M**
  - **Three topologically nontrivial choreographic motions of three bodies, Celest. Mech. Dyn. Astron. 124, 155–162 (2016)**

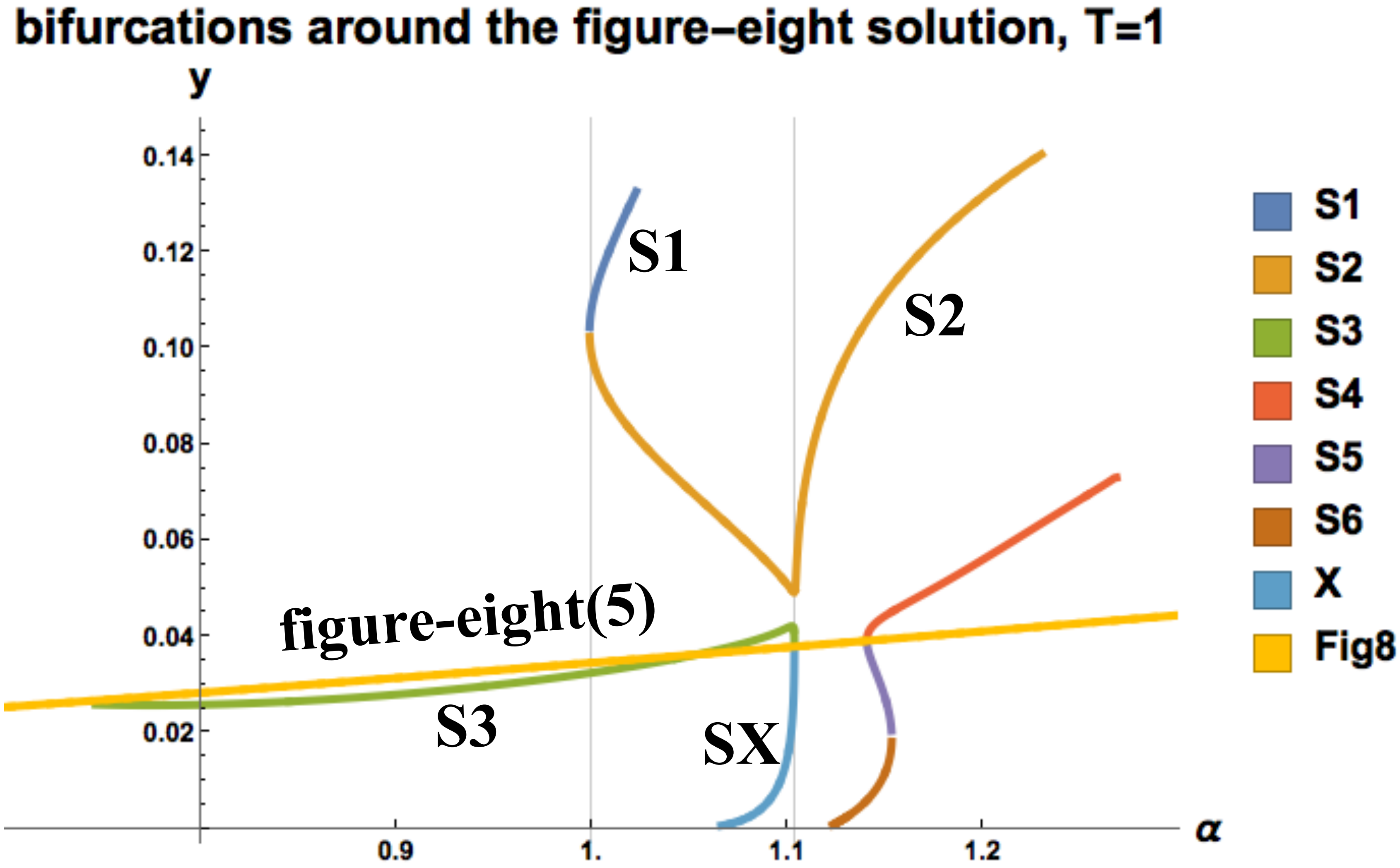
Continuation of solutions



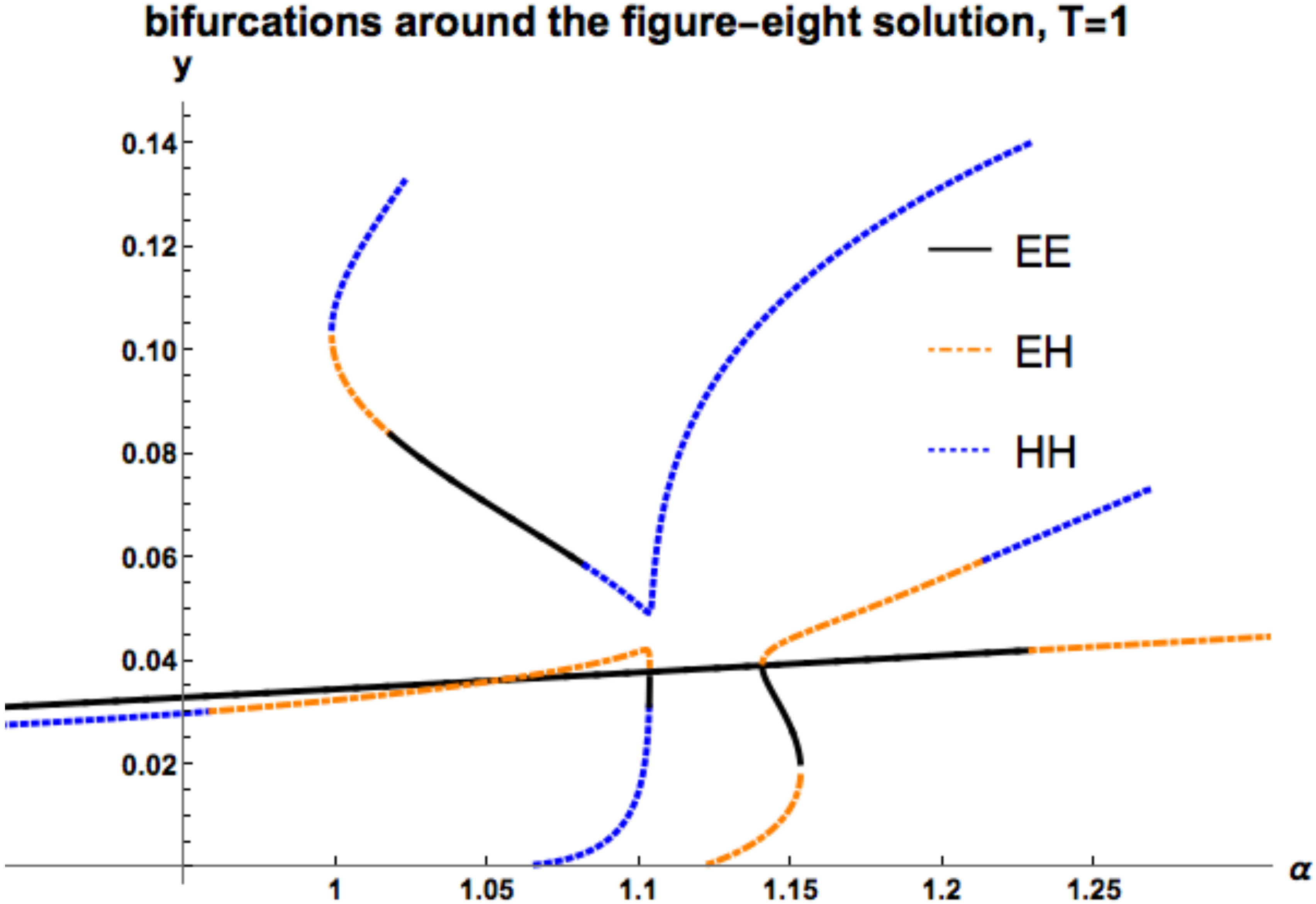
Bifurcations around figure-eight(5)



Bifurcations around figure-eight(5)



Linear stability



Hessian of action

Equal mass planar three-body problem

$$L = \frac{1}{2}\sum_{\ell}\left(\frac{dq_\ell}{dt}\right)^2 + U,\ U = \sum_{i\neq j} V(|q_i - q_j|)$$

$$S[q+\delta q] = S[q] + \delta S[q] + \frac{1}{2}\int_0^T dt \sum_{i,j} \delta q_i \underbrace{\left(-\delta_{ij}\frac{d^2}{dt^2} + \frac{\partial^2 U}{\partial q_i \partial q_j}\right)}_{=H}\delta q_j$$

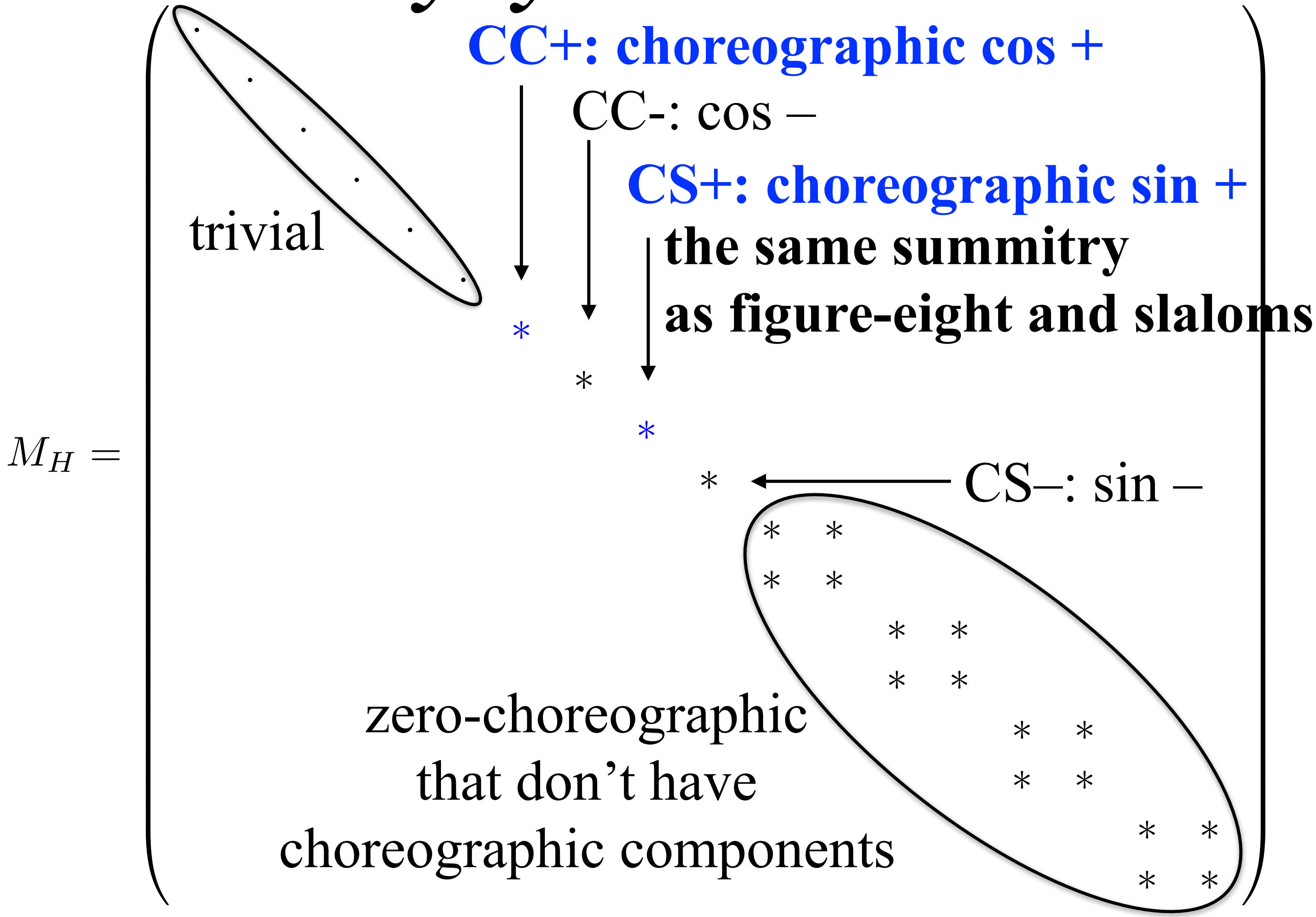
$= H$ **: Hessian**

Eigenvalue problem at a critical point $\delta S[q] = 0$

$$H\Psi = \lambda\Psi,\ \ \Psi = \begin{pmatrix}\delta q_0\\ \delta q_1\\ \delta q_2\end{pmatrix},\ \ \delta q_\ell = \begin{pmatrix}\delta q_{\ell x}\\ \delta q_{\ell y}\end{pmatrix} \in \mathbb{R}^2.$$

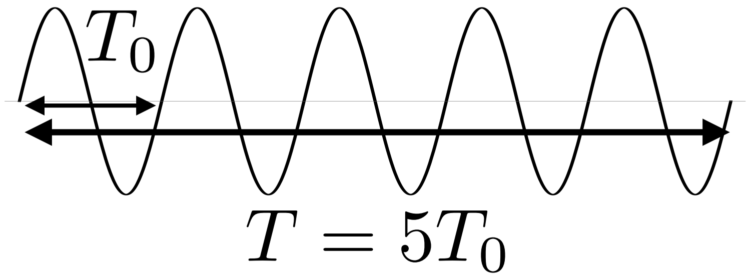
$$\delta q_\ell(t+T) = \delta q_\ell(t)$$

Decomposition of Hessian by symmetries



For detail, see appendix: "Decomposition of the Hessian matrix for action at choreographic three-body solutions" that is enclosed in the same proceedings.

Extra symmetry of Hessian for figure-eight(5)

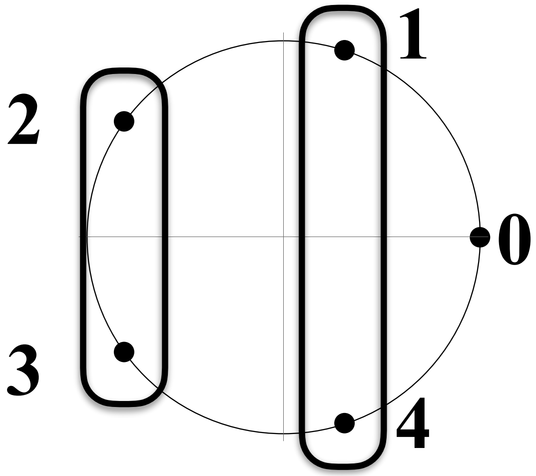


$$R^{1/5} H R^{-1/5} = H,$$

$$R^{1/5}\Psi(t) = \Psi(t + T/5)$$

$$R^{1/5}\Psi(t) = e^{2n\pi i/5}\Psi(t)$$

eigenfunctions are labeled by n=0,1,2,3,4



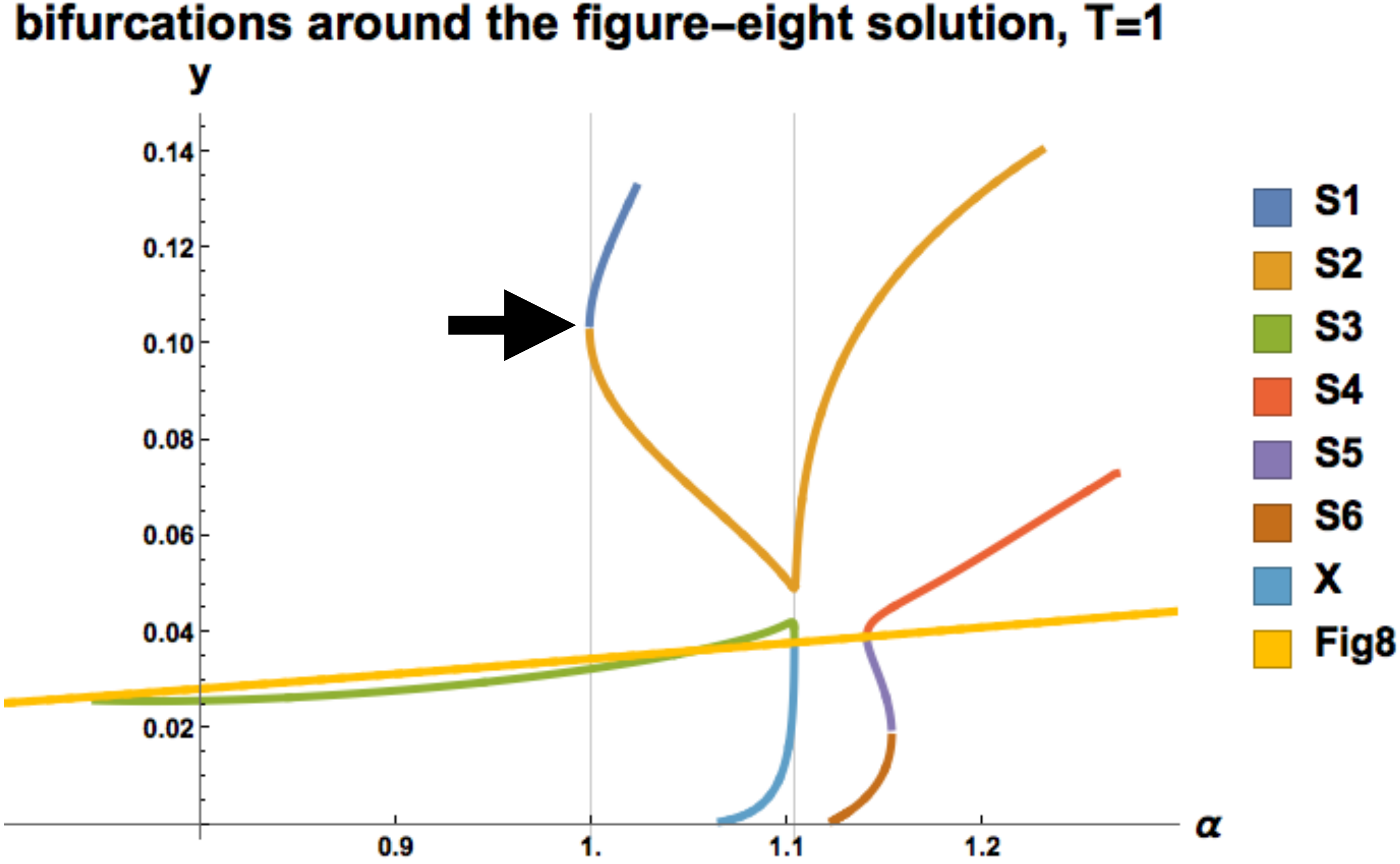
$n = 0$: singlet

$n = 1, 4$: degenerated doublet

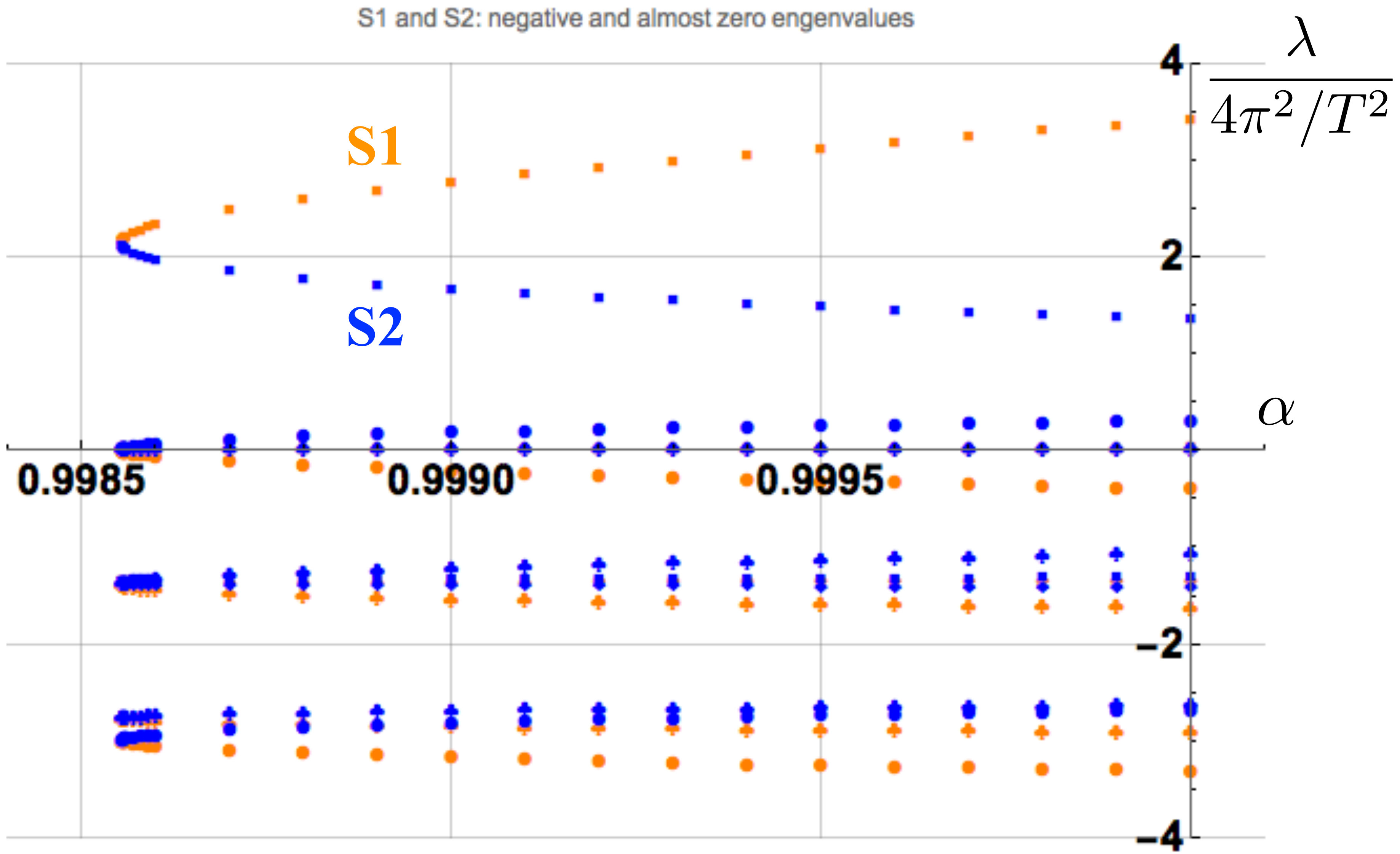
$n = 2, 3$: degenerated doublet

Saddle-node bifurcation

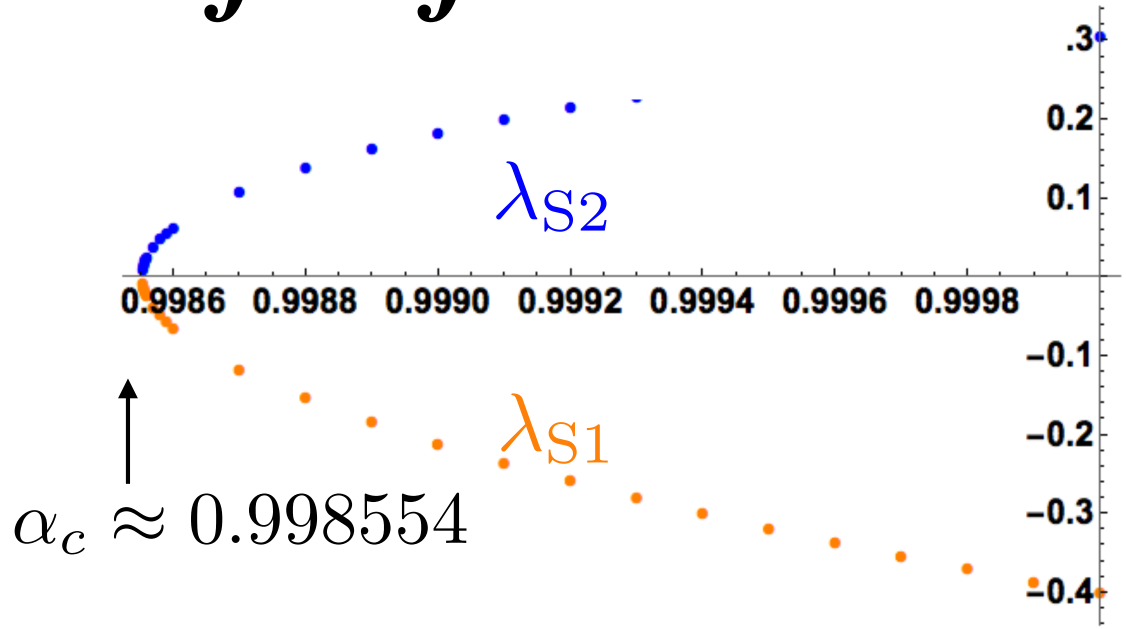
$$\alpha \sim 0.998554$$

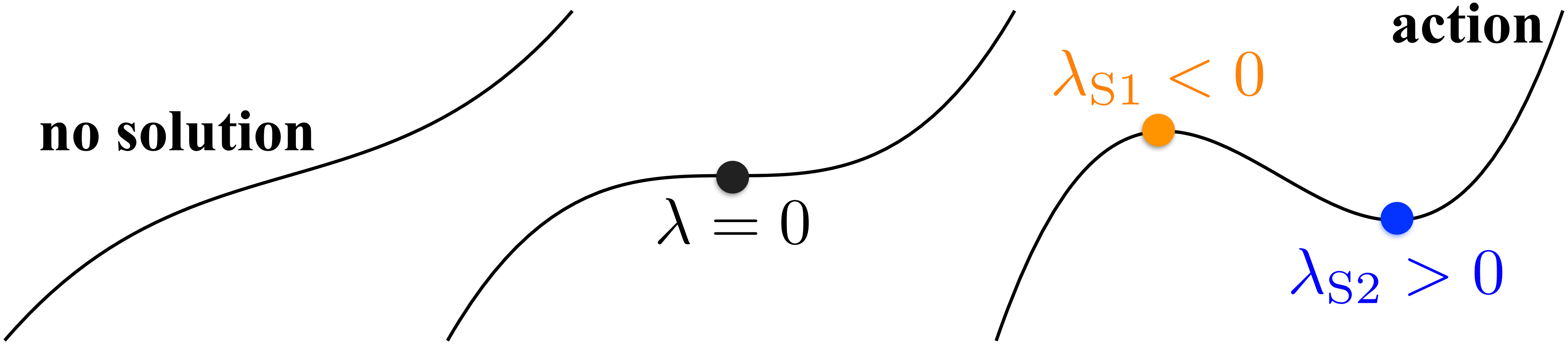


Eigenvalues of H



Eigenvalues of H for CS+

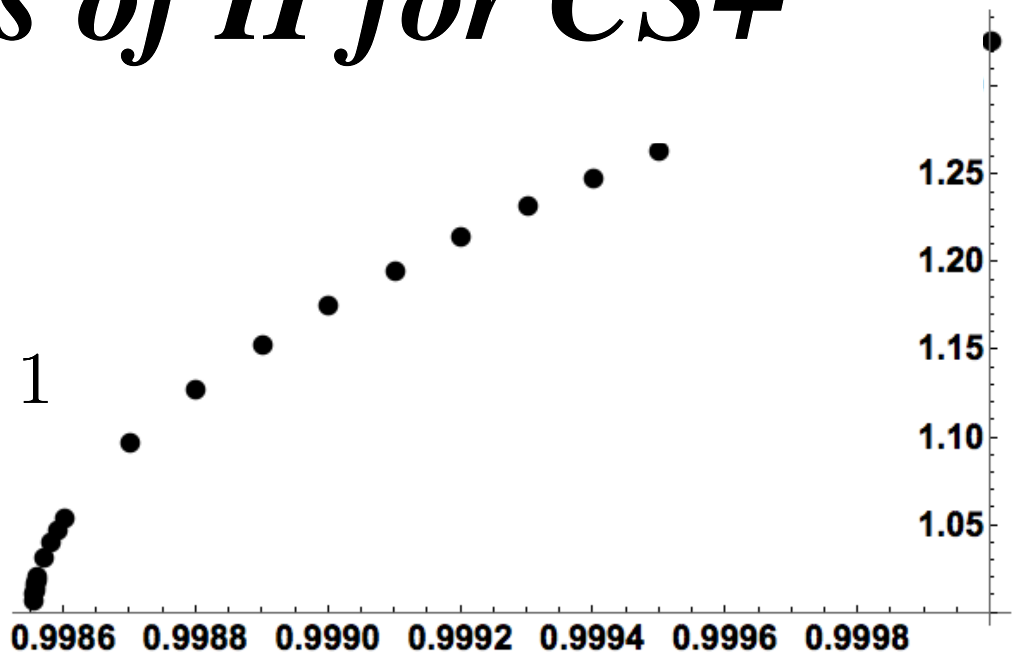




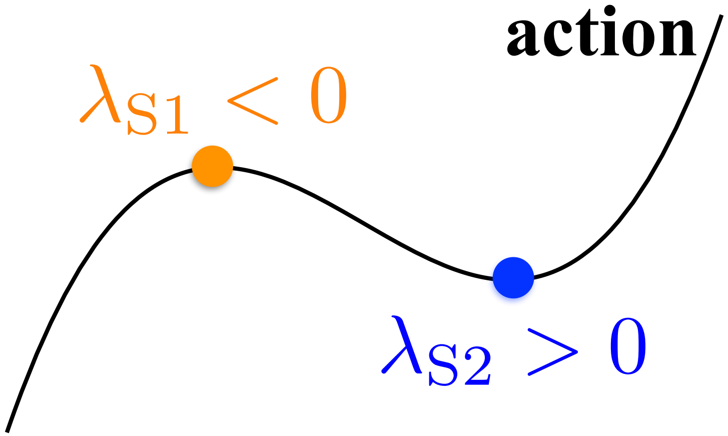
Euler characteristic: $0 = (-1)^1 + (-1)^0$

Eigenvalues of H for CS+

$$-\frac{\lambda_{S1}}{\lambda_{S2}} \to 1$$



$$S'(x) = a(x - x_1)(x - x_2) \Rightarrow S''(x_1) + S''(x_2) = 0$$



Euler characteristic

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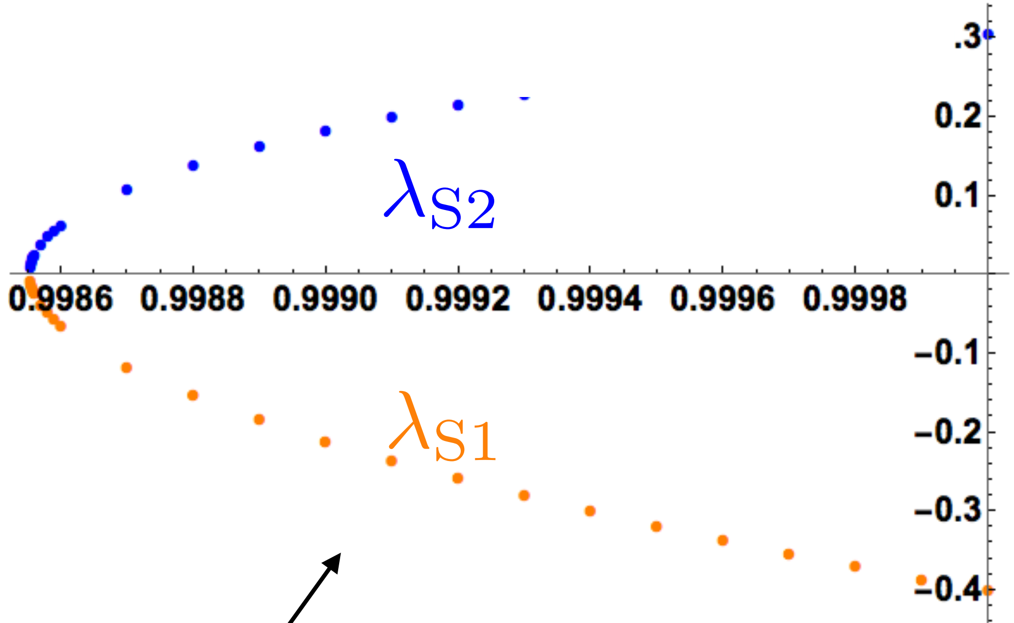
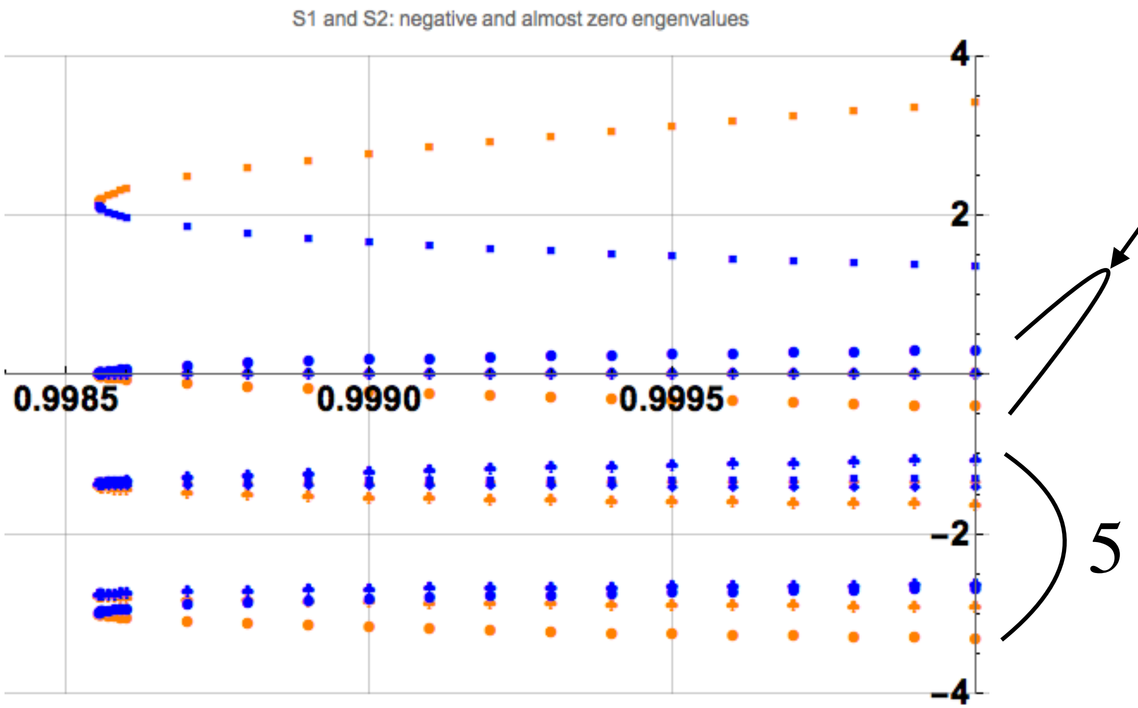
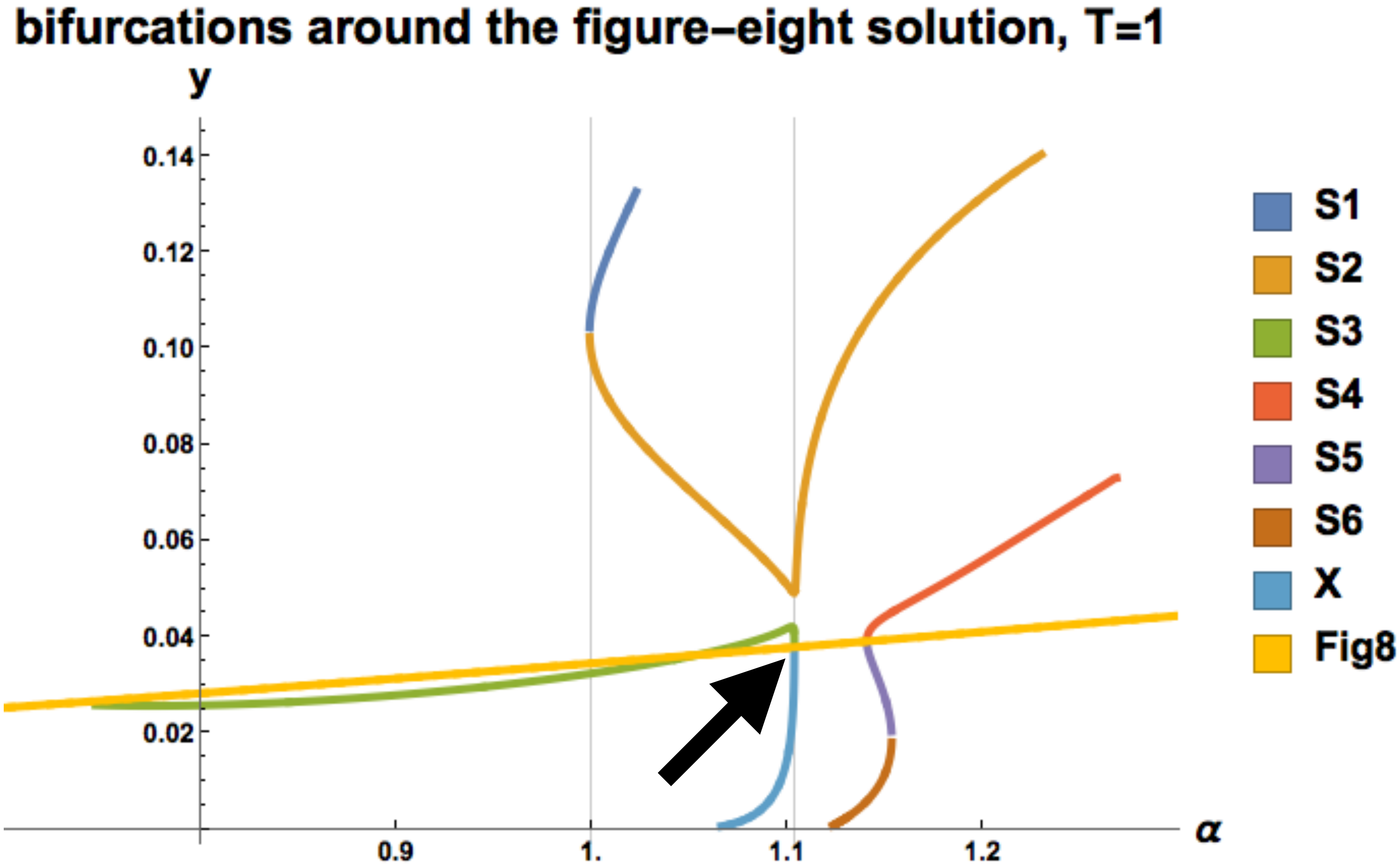




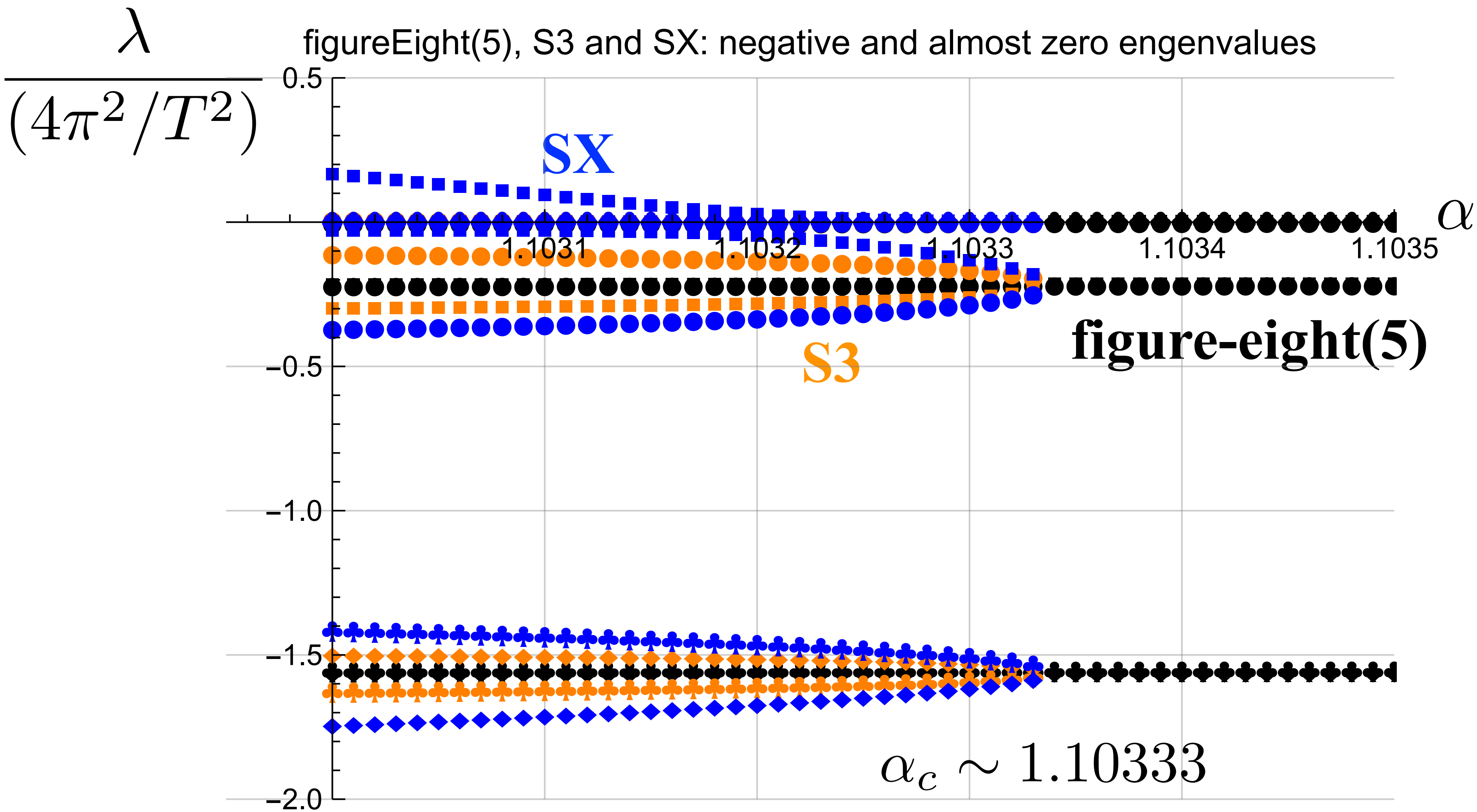
$$0 = (-1)^5\Big((-1)^1 + (-1)^0\Big)$$

Pitchfork bifurcation

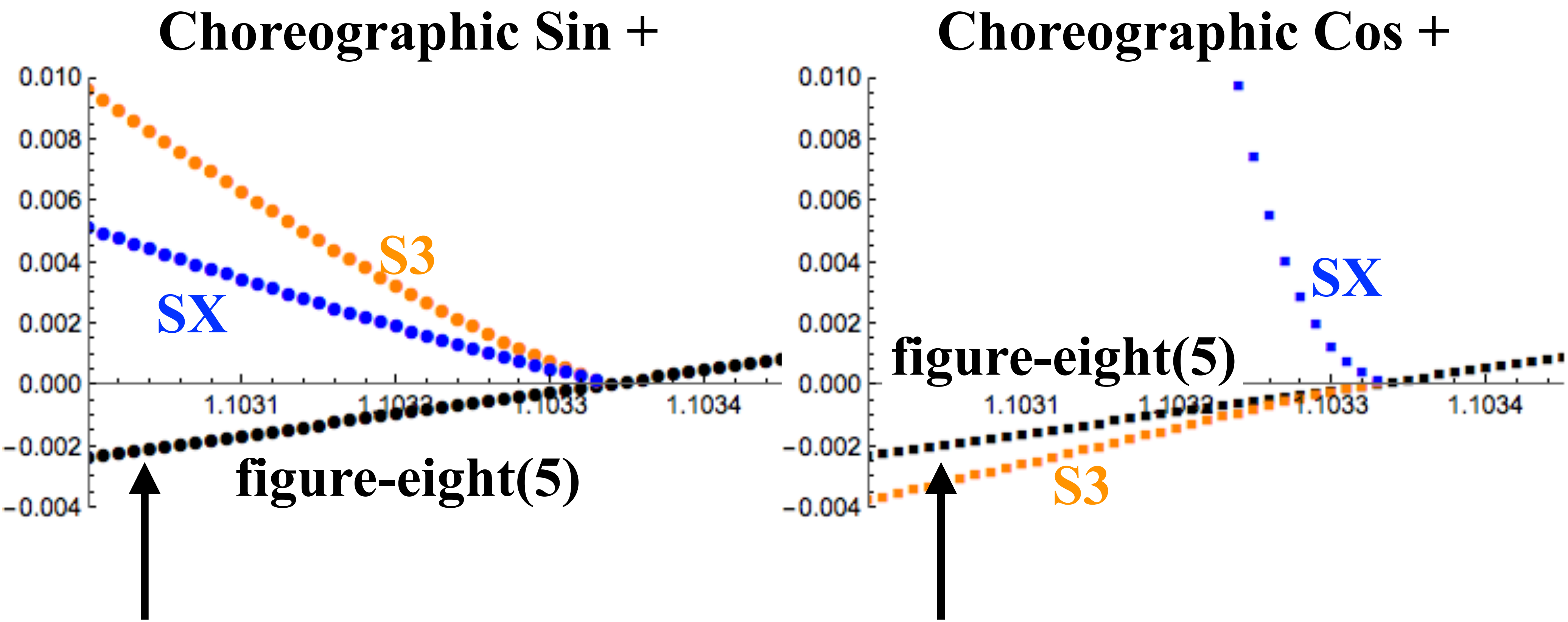
$$\alpha \sim 1.10333$$



Eigenvalues of H

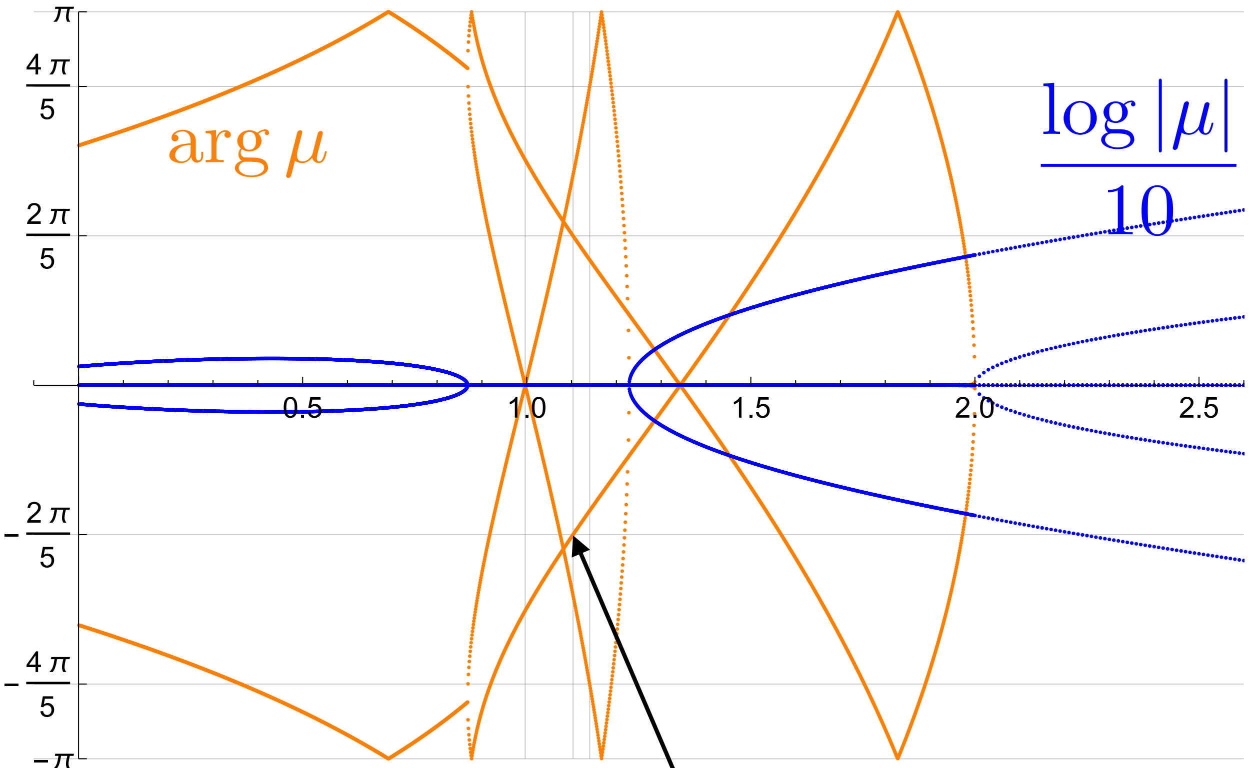


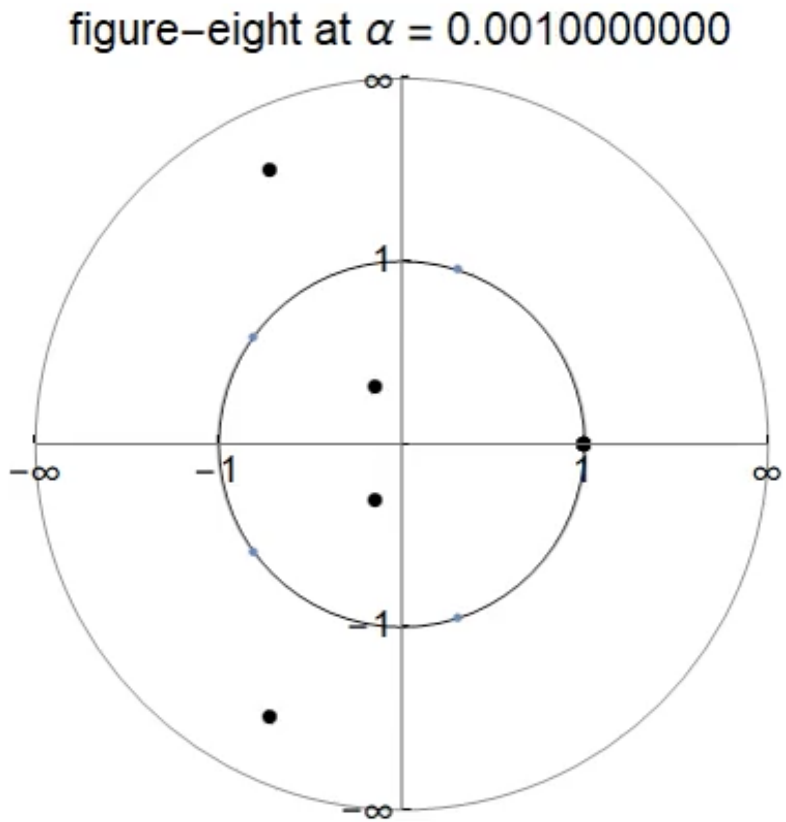
Eigenvalues of H for CS+, CC+



the eigenvalues for CS+ and CC+ of figure-eight(5) shown here are degenerated (1-4 doublet)

linear stability of figure-eight(1)

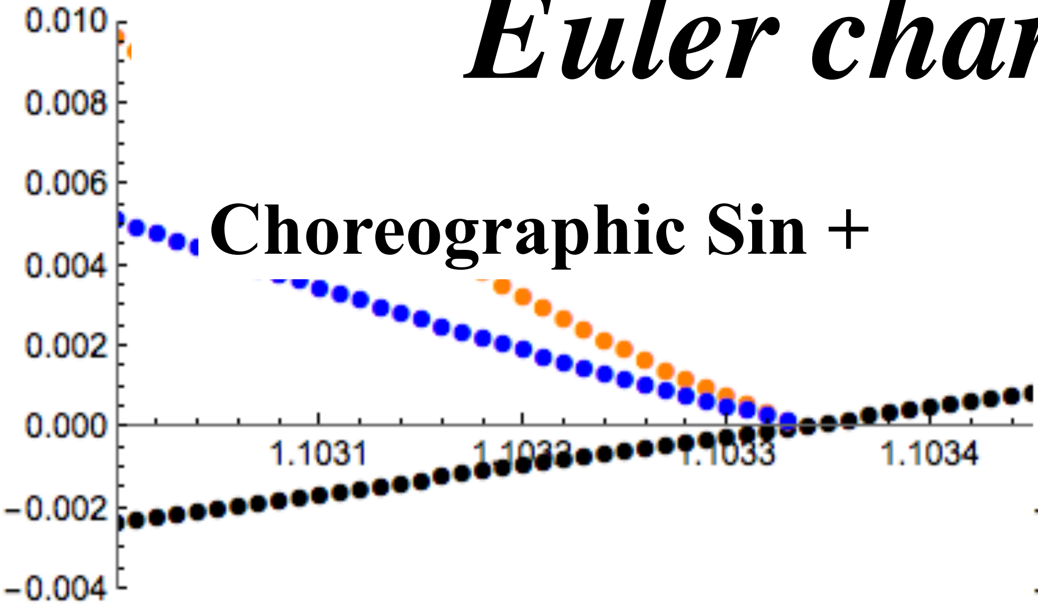


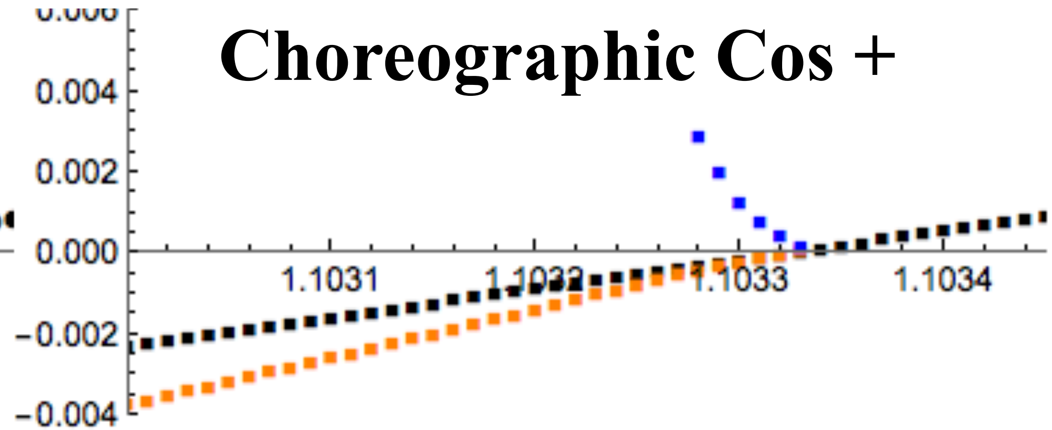


period-5 bifurcation: S3, SX

choreographic sin+ and cos+
(1-4 doublet)

Euler characteristics

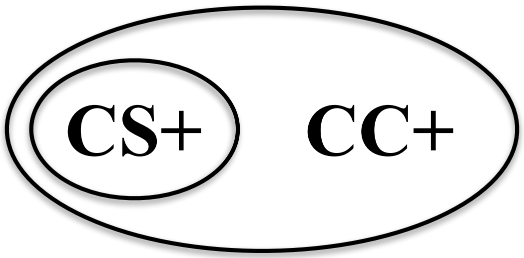




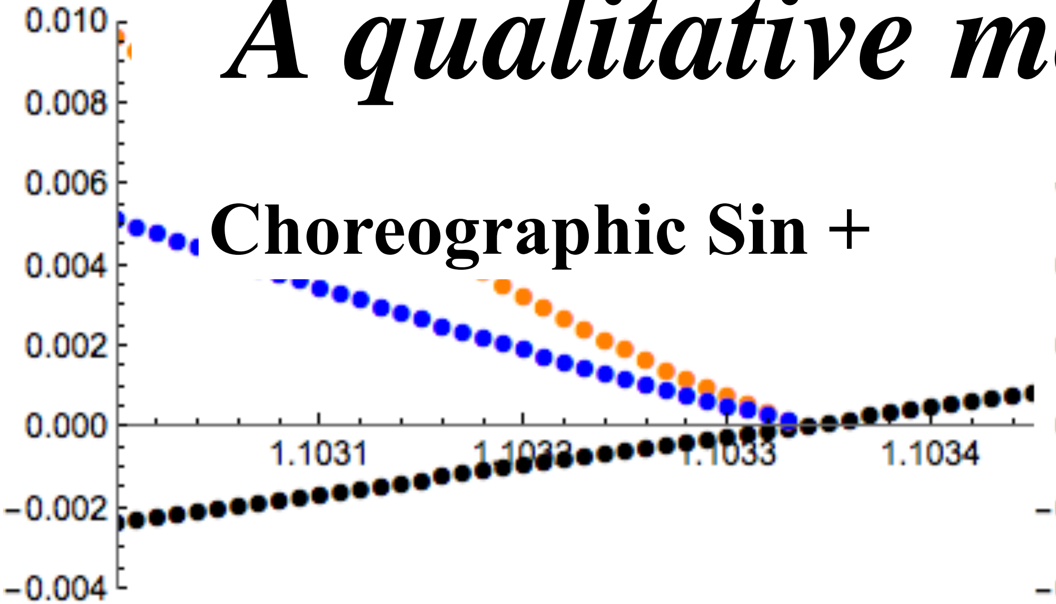
CS+ : $(-1)^1 + (-1)^0 + (-1)^0 = -1+1+1 = 1 = (-1)^0$

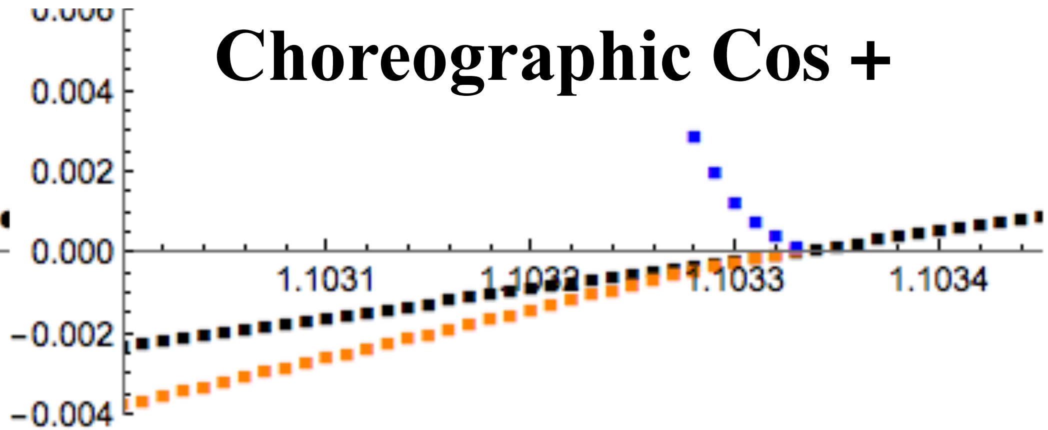
CC+ : $(-1)^1 + (-1)^0 + (-1)^1 = -1+1-1 = -1 \neq (-1)^0$

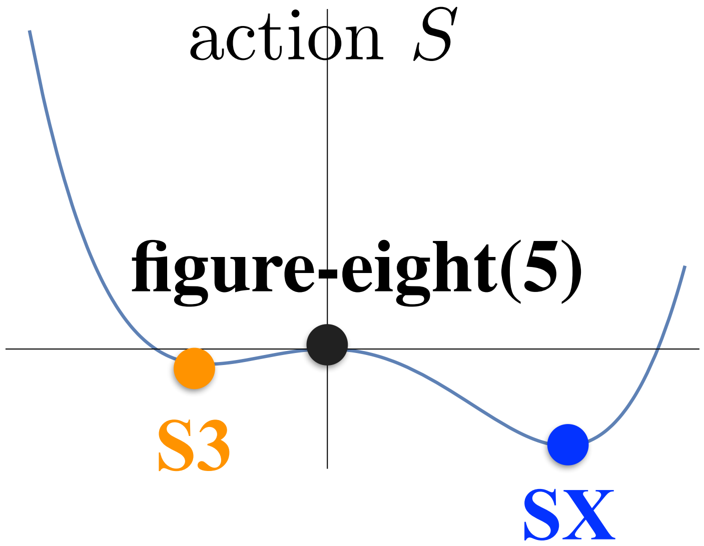
CS+ and CC+ : $(-1)^2 + (-1)^0 + (-1)^1 = 1+1-1 = 1 = (-1)^0$

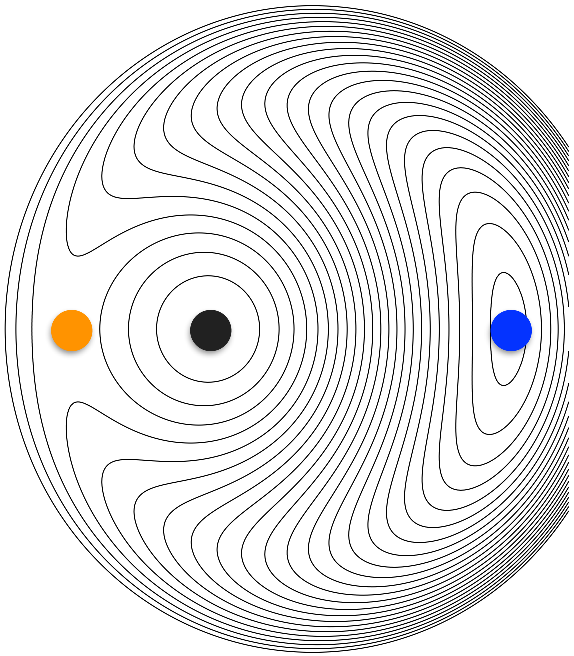


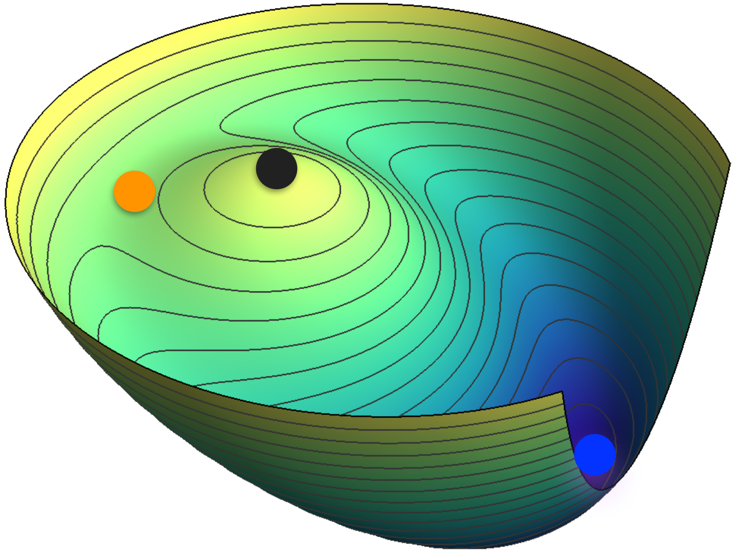
A qualitative model for action











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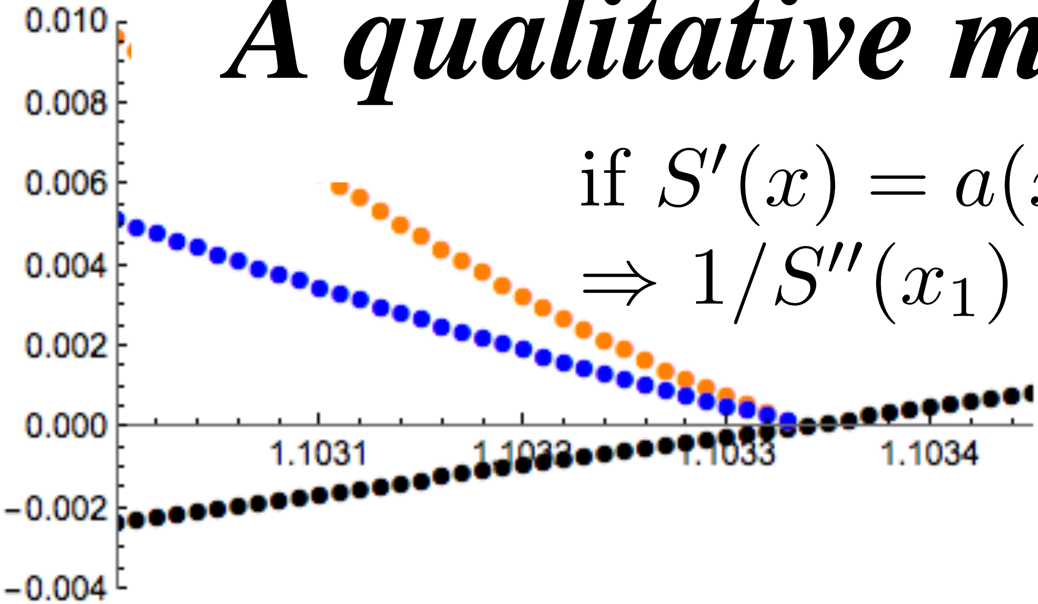
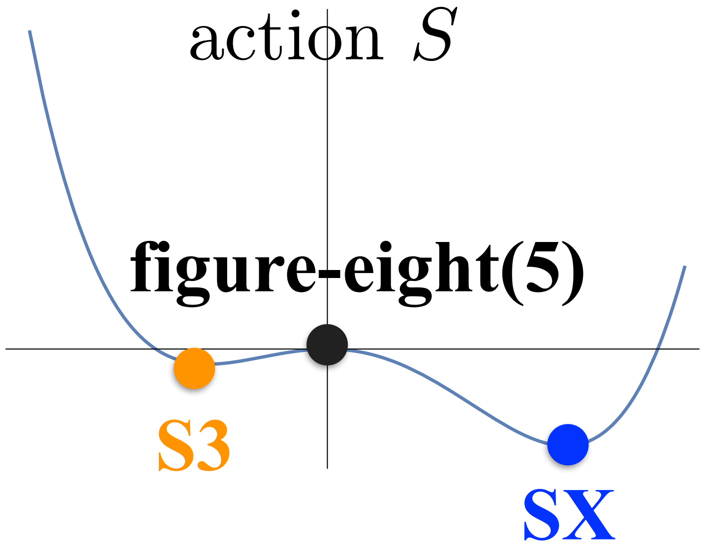
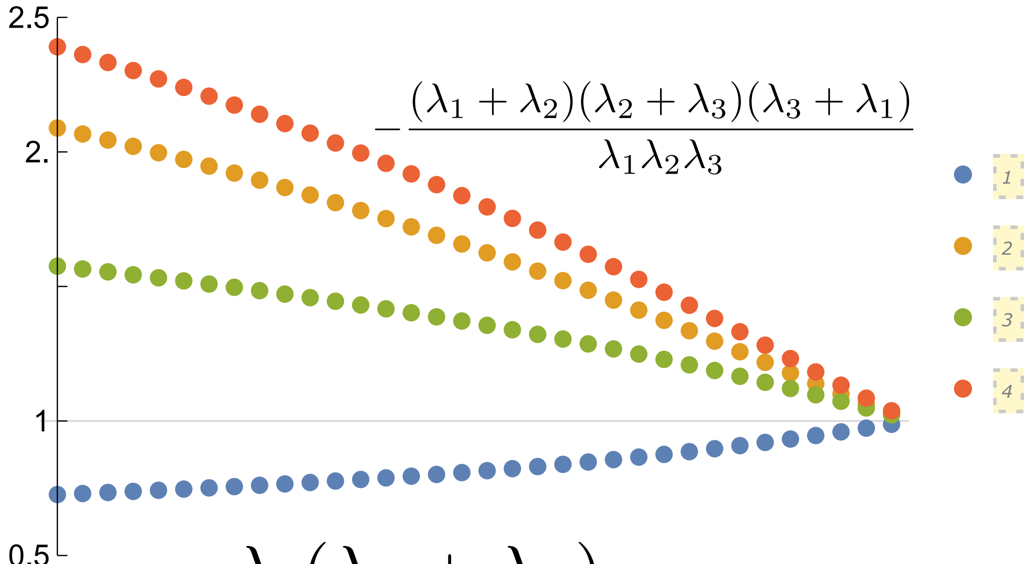


$$\text{if } S'(x) = a(x - x_1)(x - x_2)(x - x_3)$$
$$\Rightarrow 1/S''(x_1) + 1/S''(x_2) + 1/S''(x_3) = 0$$

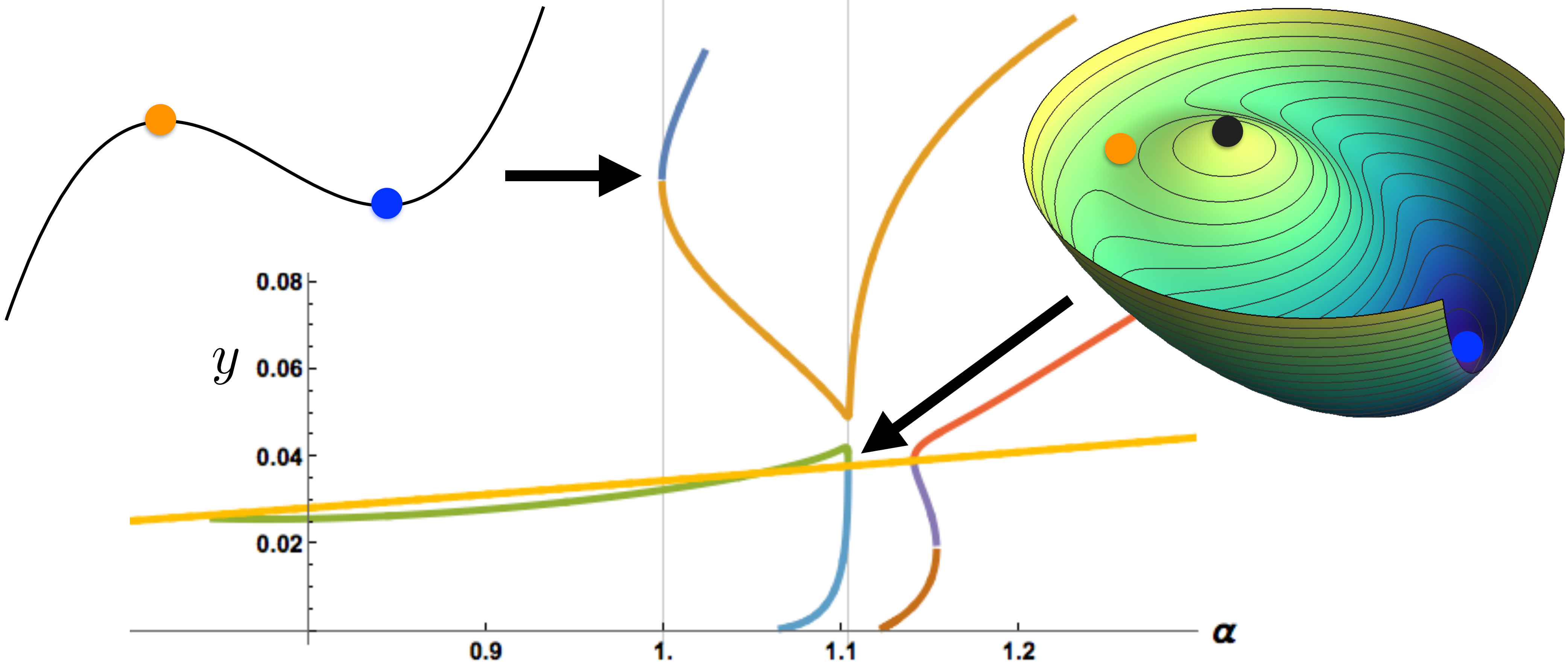
actually $\sum \lambda_k^{-1} \sim 200 \neq 0$



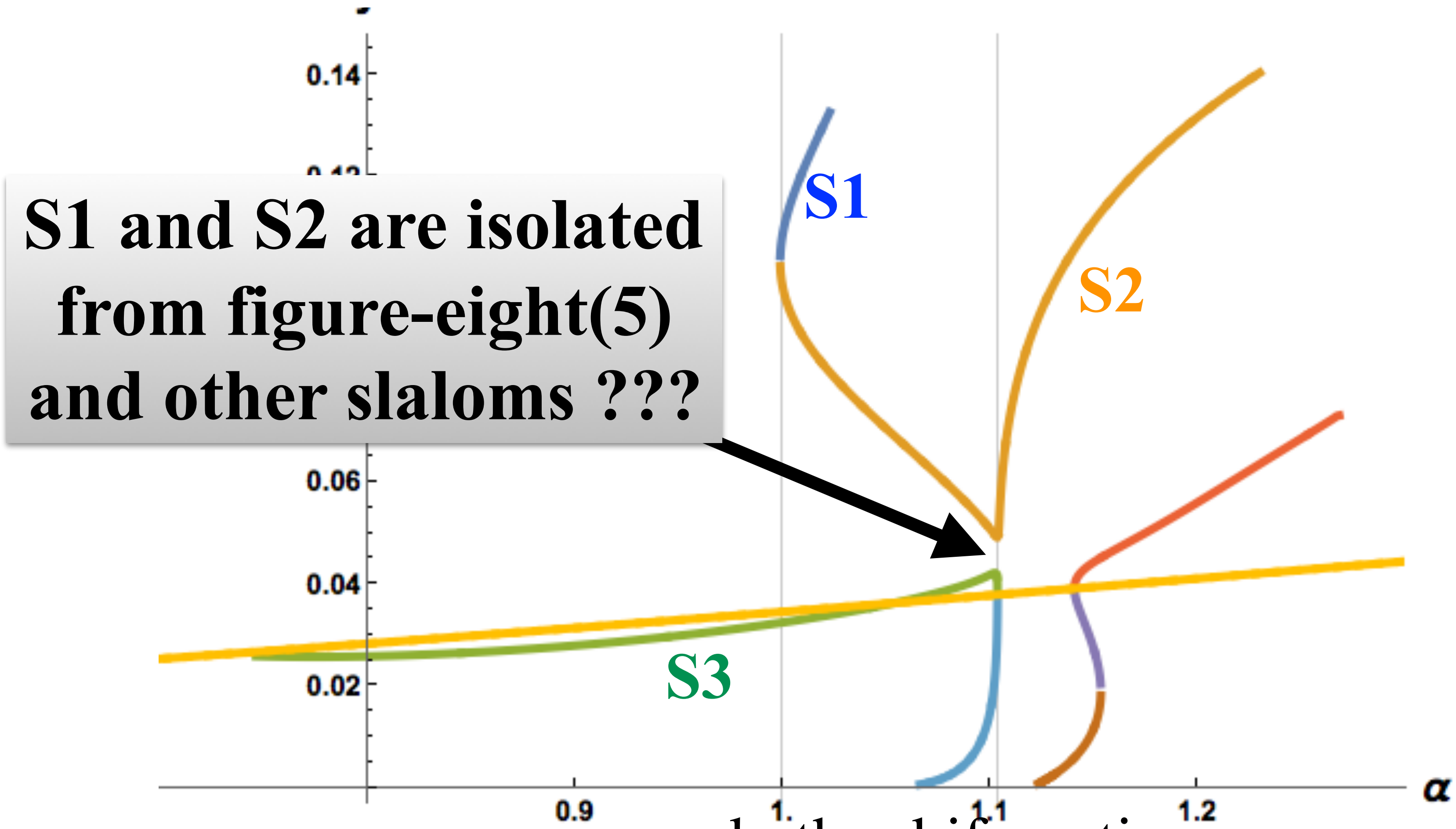


$$-\frac{\lambda_i(\lambda_j + \lambda_k)}{\lambda_j \lambda_k} \to 1$$

Bifurcations around figure-eight(5)



Bifurcations around figure-eight(5)



… and other bifurcations

period doubling, non-figure-eight, non-choreographic, other periods (7,11,…), …