三体8の字解の分岐
作用と線形安定性の視点から

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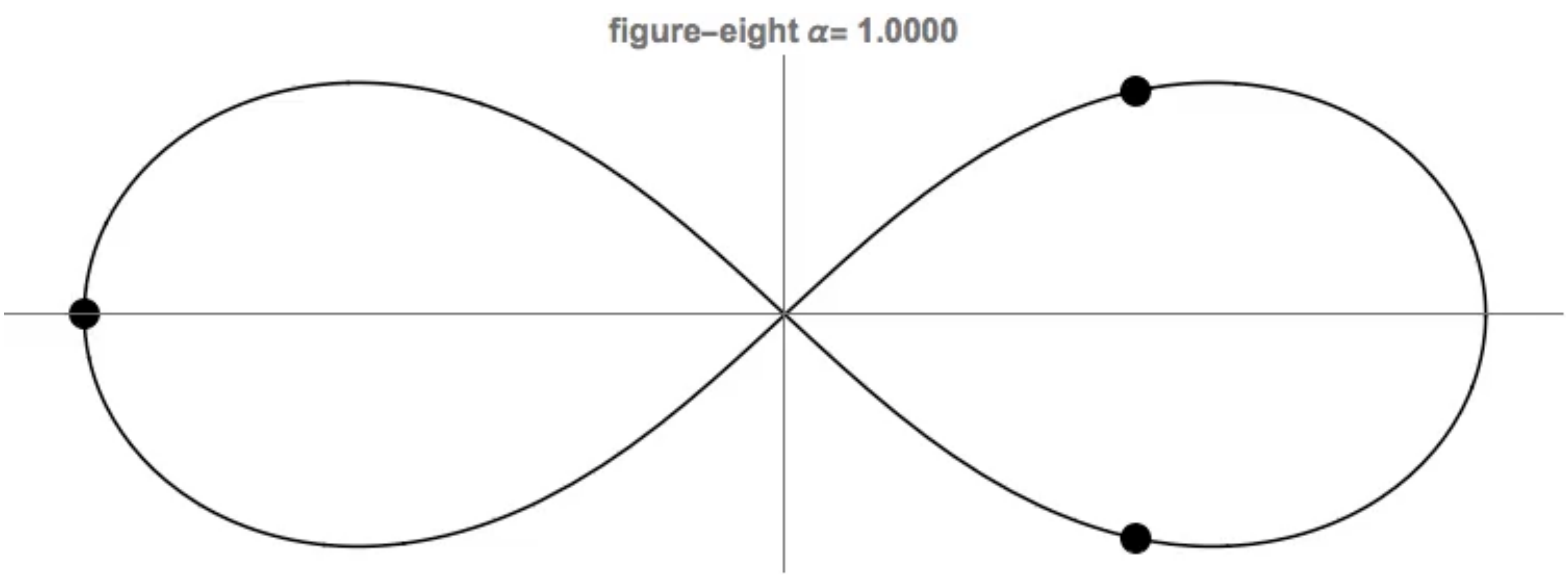
figure-eight solution

Lagrangian: $L = \sum_k \left| \frac{dq_k}{dt} \right|^2 + U(q), \ q_k \in \mathbb{R}^2$

Potential: $U(q) = \begin{cases} \alpha^{-1} \sum_{ij} 1/r_{ij}^{\alpha} & \text{for } \alpha \neq 0, \\ \sum_{ij} \log |r_{ij}| & \text{for } \alpha = 0. \end{cases}$ $\quad -2 < \alpha$



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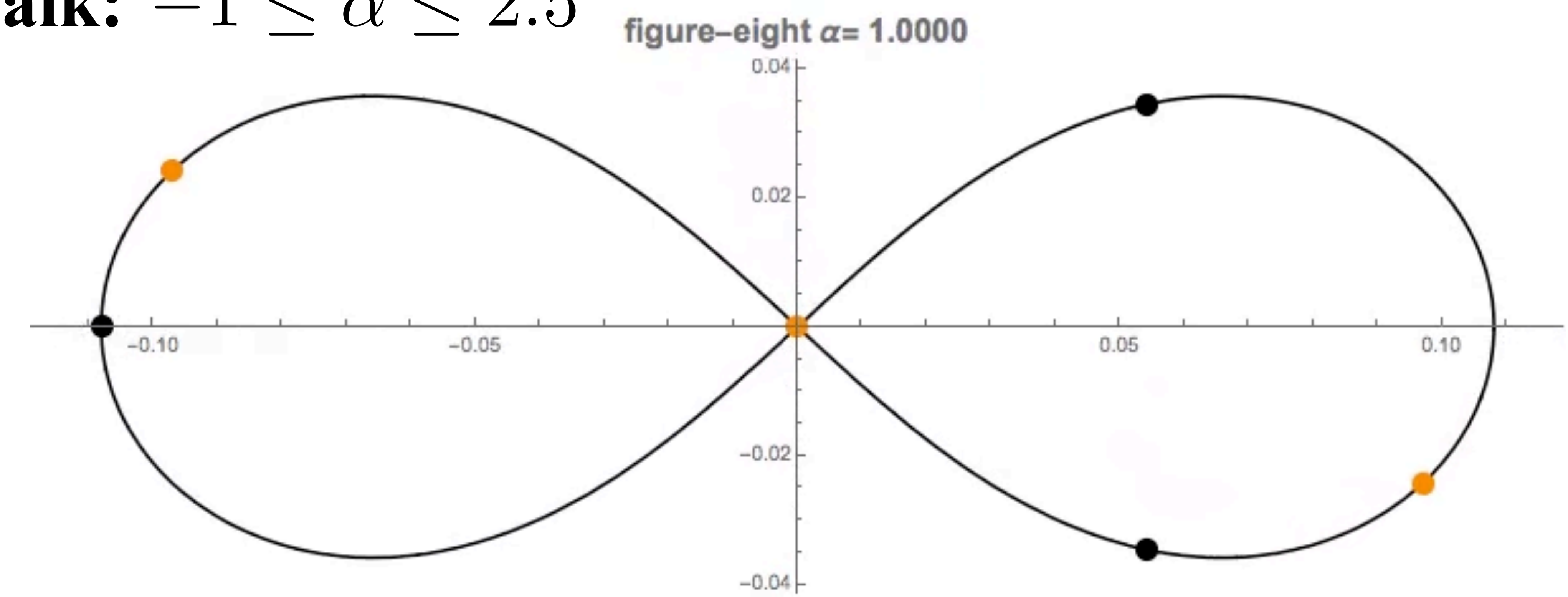
this talk: $-1 \leq \alpha \leq 2.5$



second derivative of action

$$S[q] = \int dt \left(\frac{1}{2} \left| \frac{dq_k}{dt} \right|^2 + U(q) \right)$$

$q(t)$: figure-eight solution

$$S[q + \delta q] = S[q] + \int dt \ \delta q \left(-\frac{d^2 q}{dt^2} + U_q(q) \right)$$
$$+ \frac{1}{2} \int dt \ \delta q \left(-\frac{d^2}{dt^2} + U_{qq}(q) \right) \delta q$$

eigenvalue problem

$$\left(-\frac{d^2}{dt^2} + U_{qq}(q) \right) \psi(t) = \lambda \ \psi(t), \ \ \psi(t+T) = \psi(t)$$

condition for bifurcation

equation of motion

$$-\frac{d^2q}{dt^2} + U_q(q) = 0$$ **: figure-eight solution**

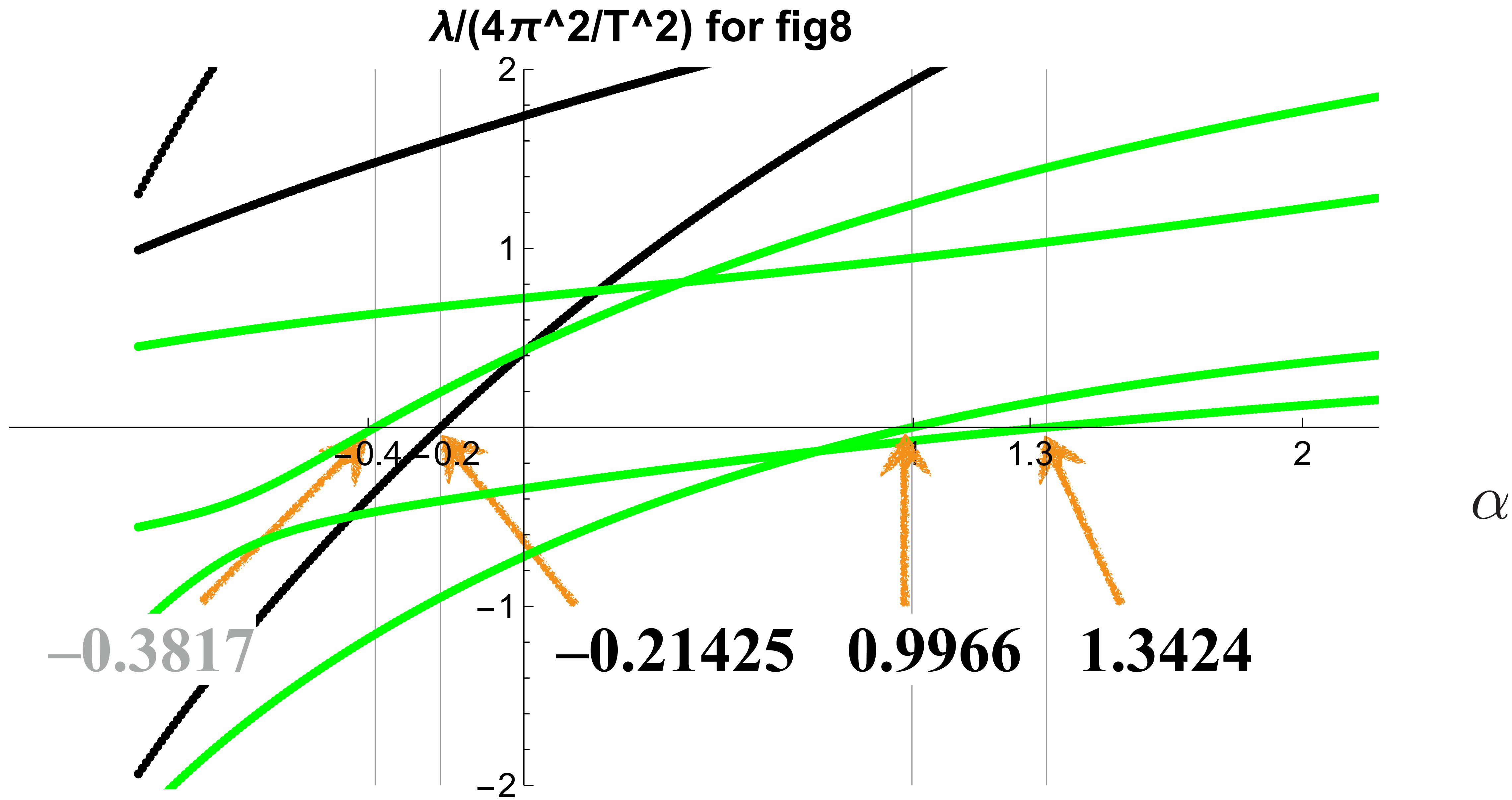
$$-\frac{d^2(q+\delta q)}{dt^2} + U_q(q+\delta q) = 0$$ **: a bifurcated solution**

$$\Longrightarrow \quad \left(-\frac{d^2}{dt^2} + U_{qq}(q)\right)\delta q = 0$$

$\lambda \to 0$ **:a necessary condition**

$$\left(-\frac{d^2}{dt^2} + U_{qq}(q)\right)\psi(t) = \lambda\ \psi(t),\ \ \psi(t+T) = \psi(t)$$

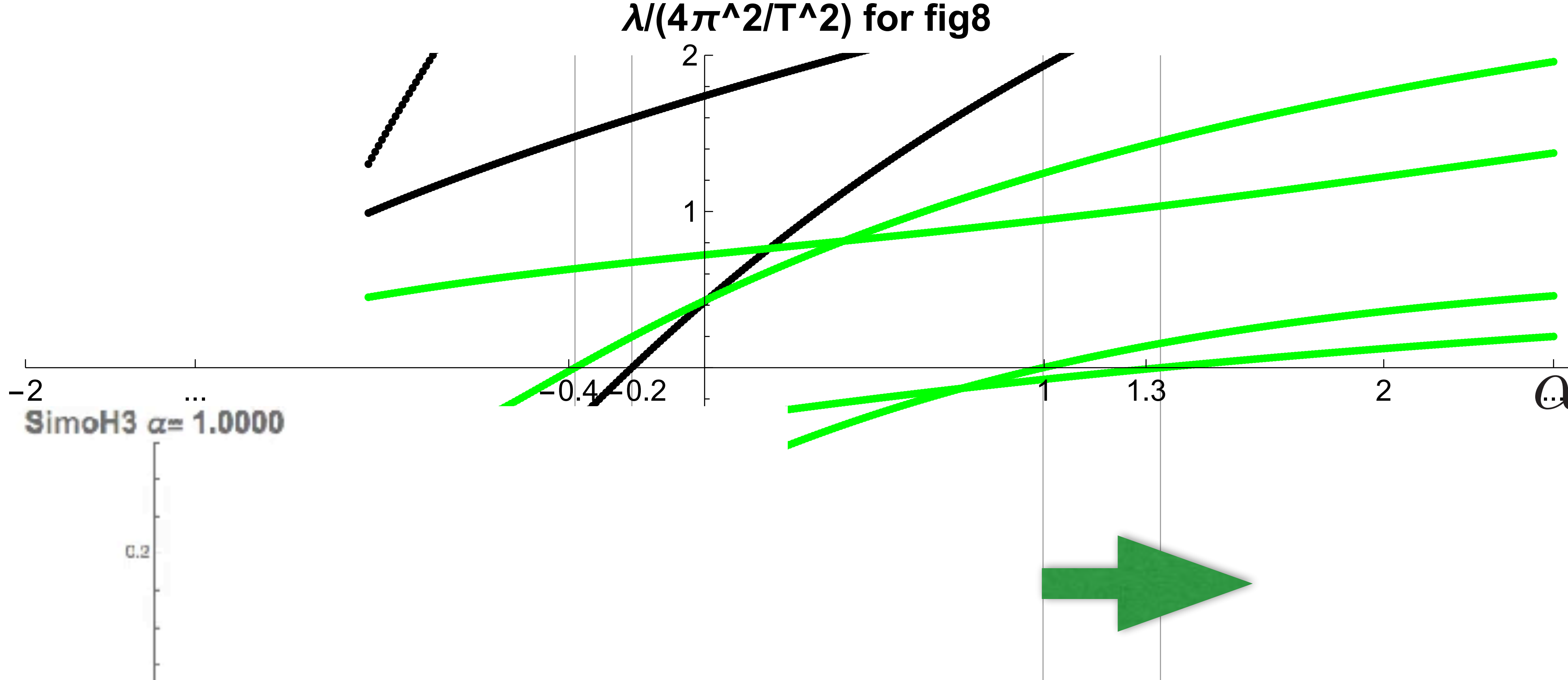
Zero eigenvalues for the figure-eight and Bifurcations

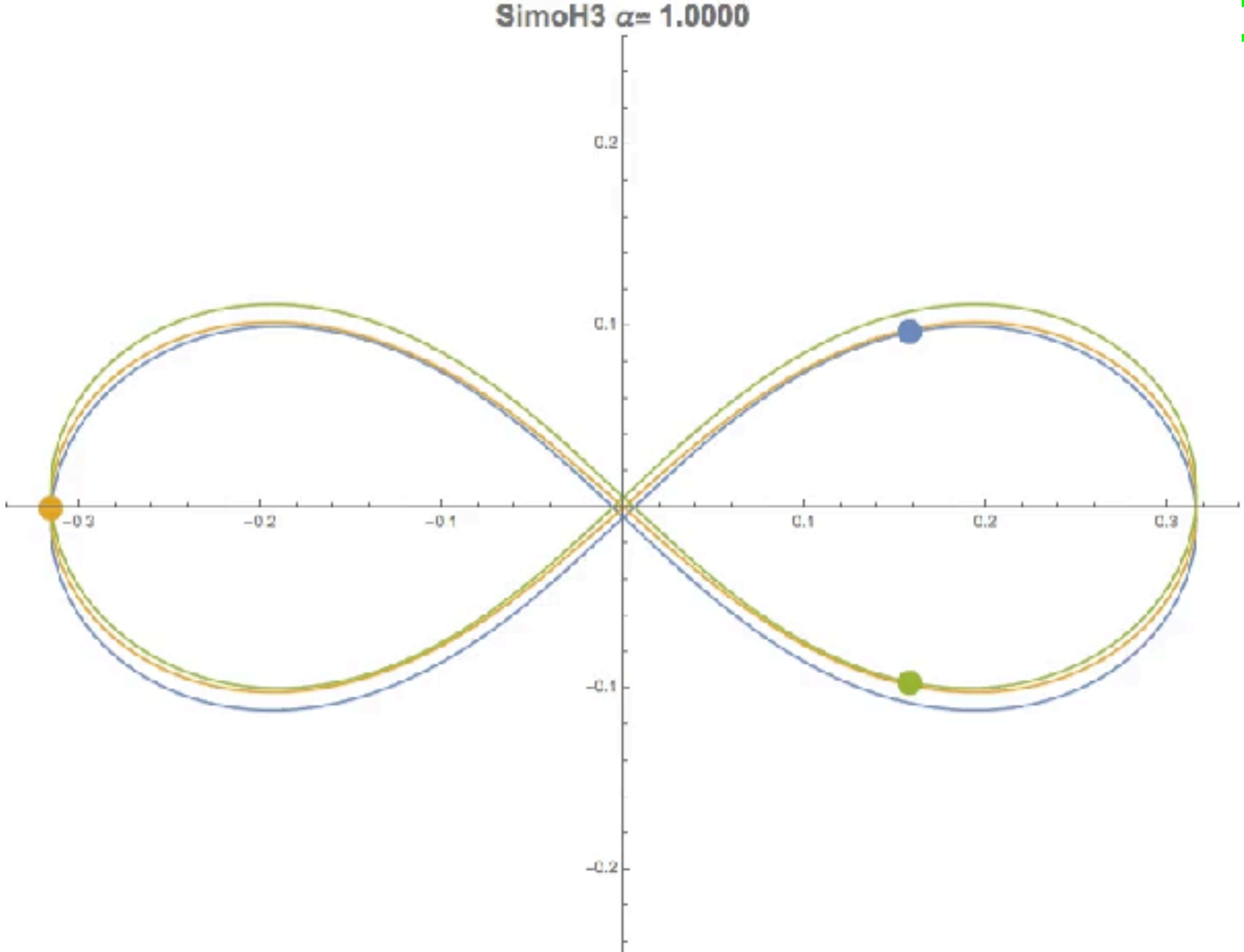


not degenerated: choreographic

doubly degenerated: "zero-choreographic"

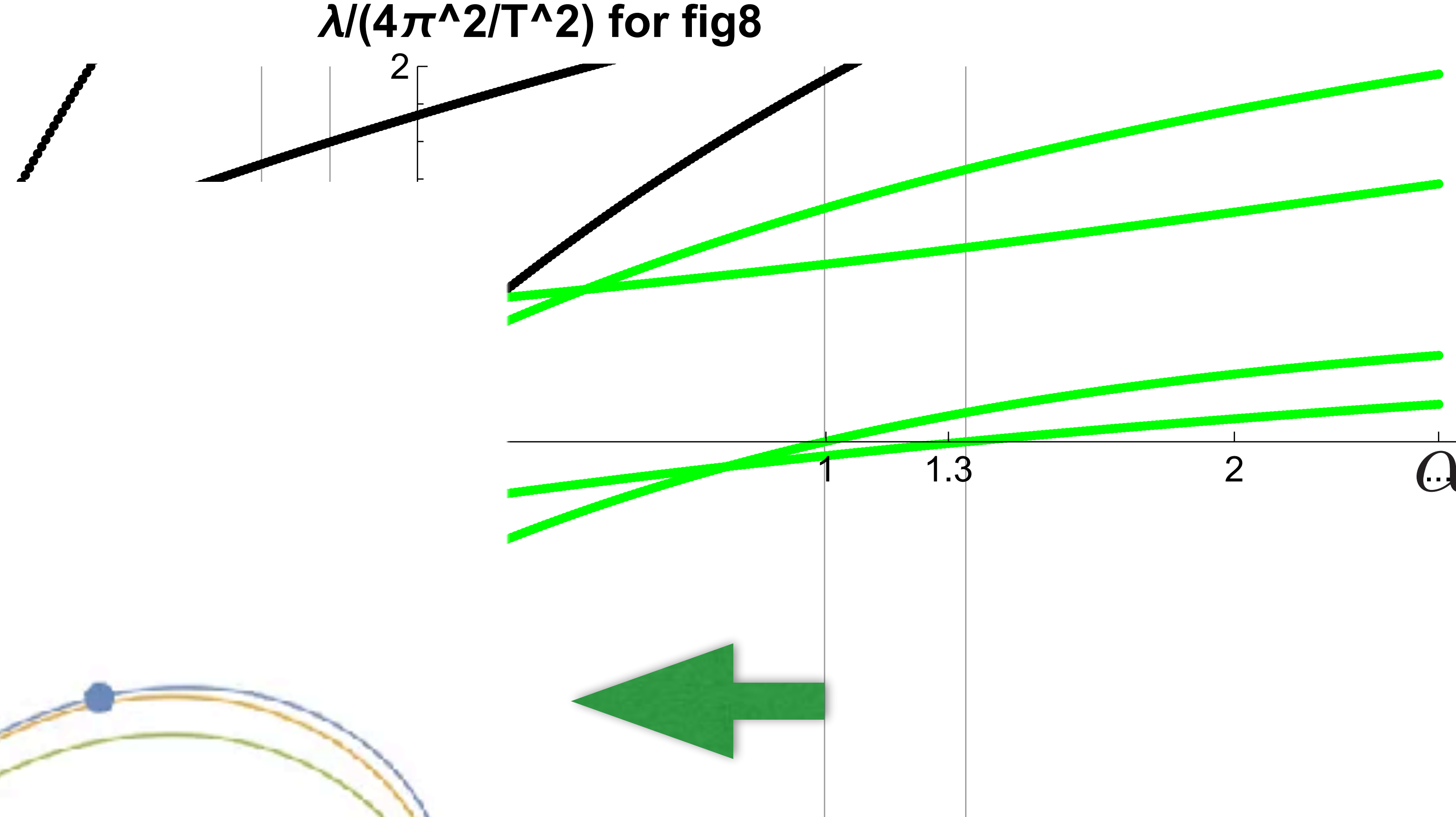
bifurcations from figure-eight

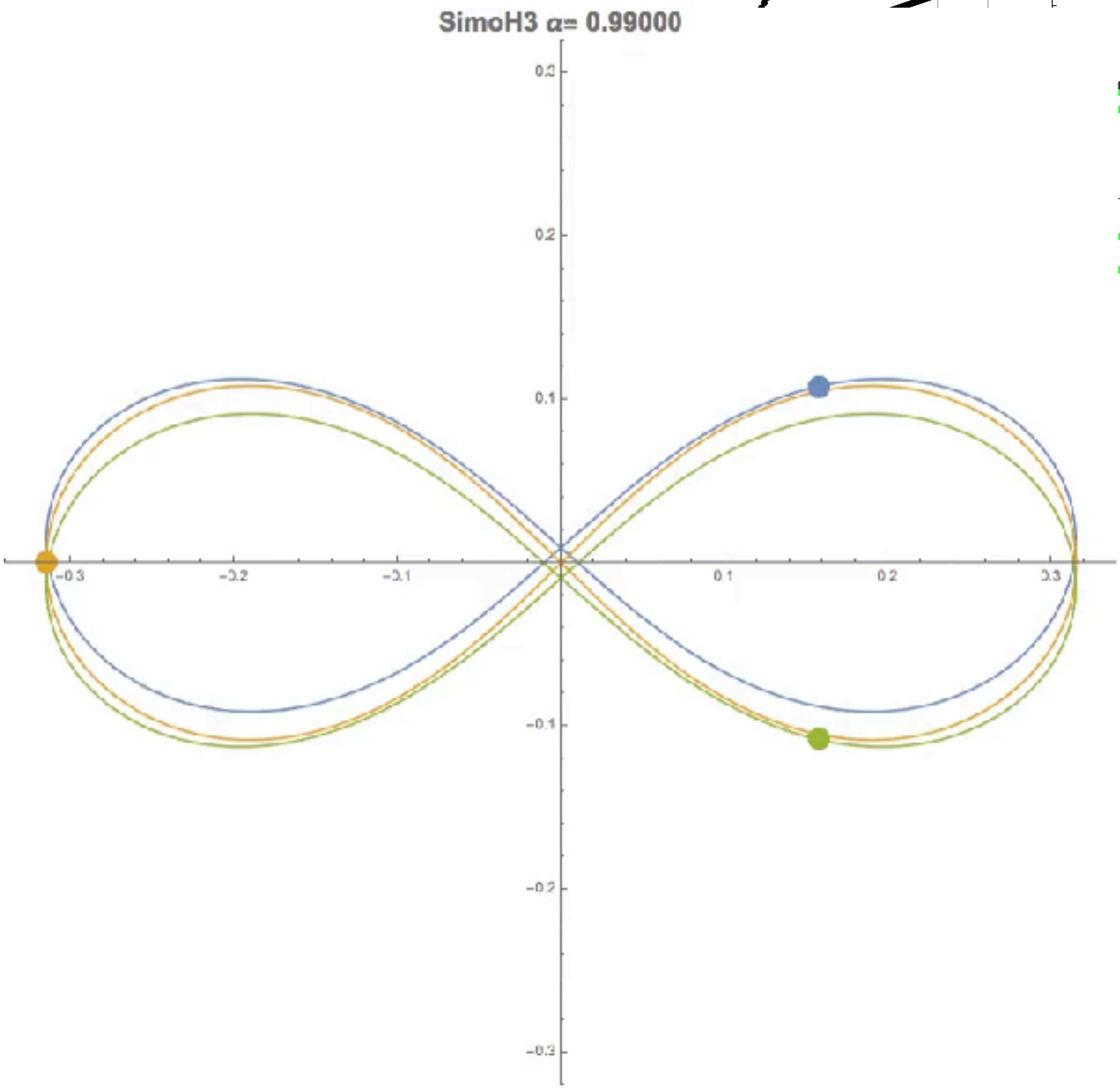




Simo H

bifurcations from figure-eight





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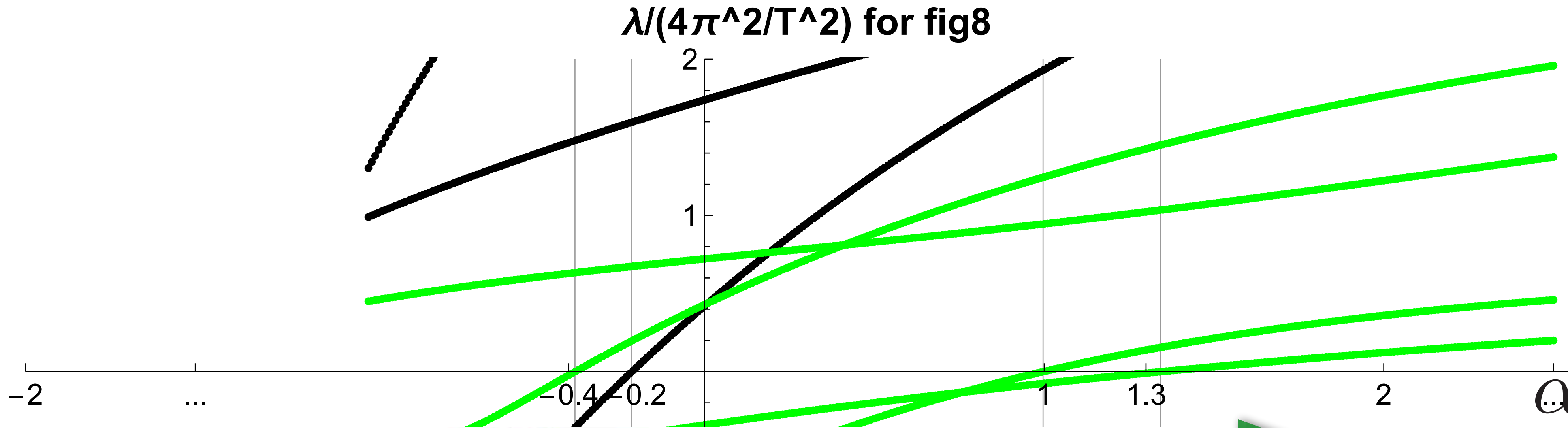
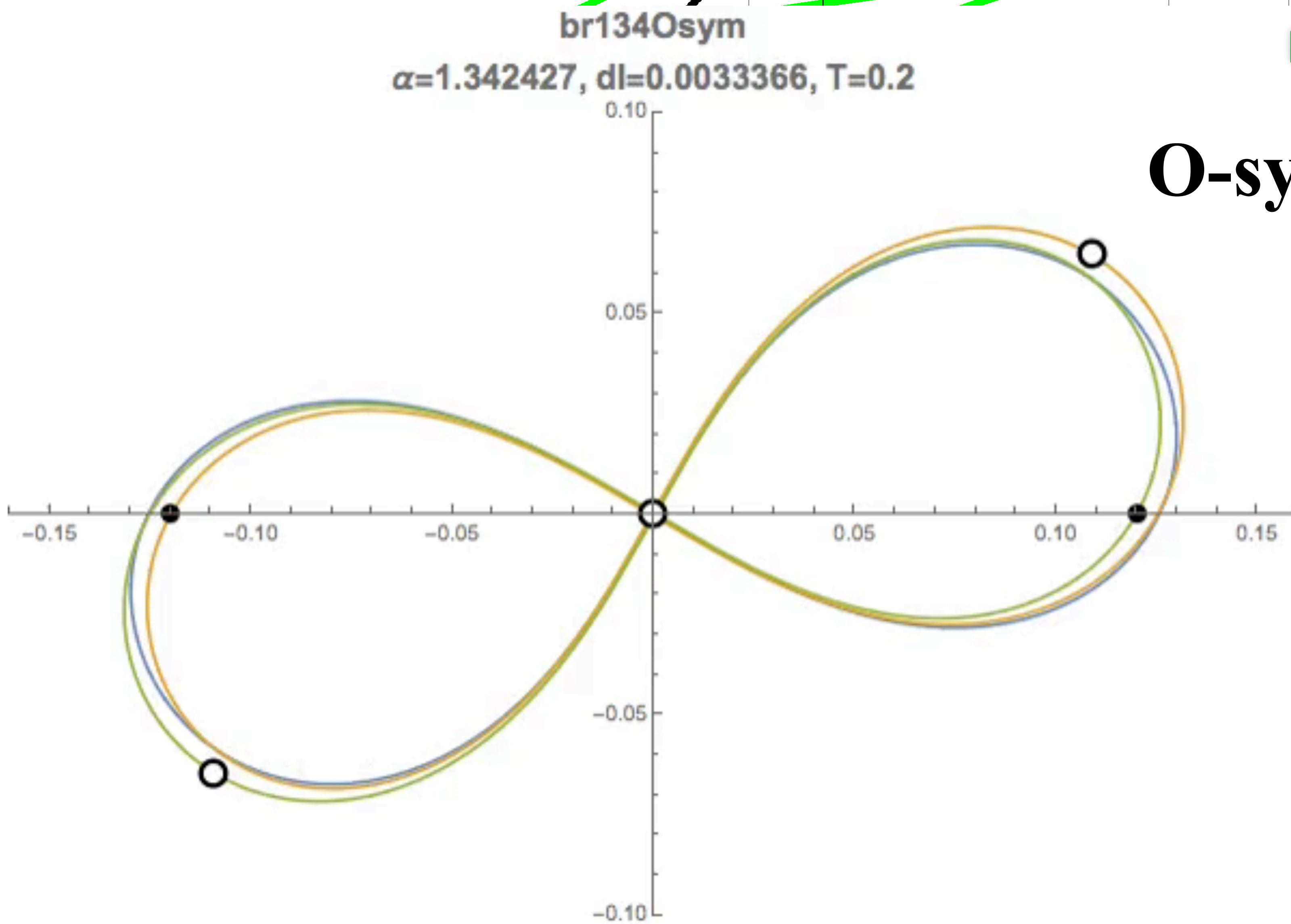
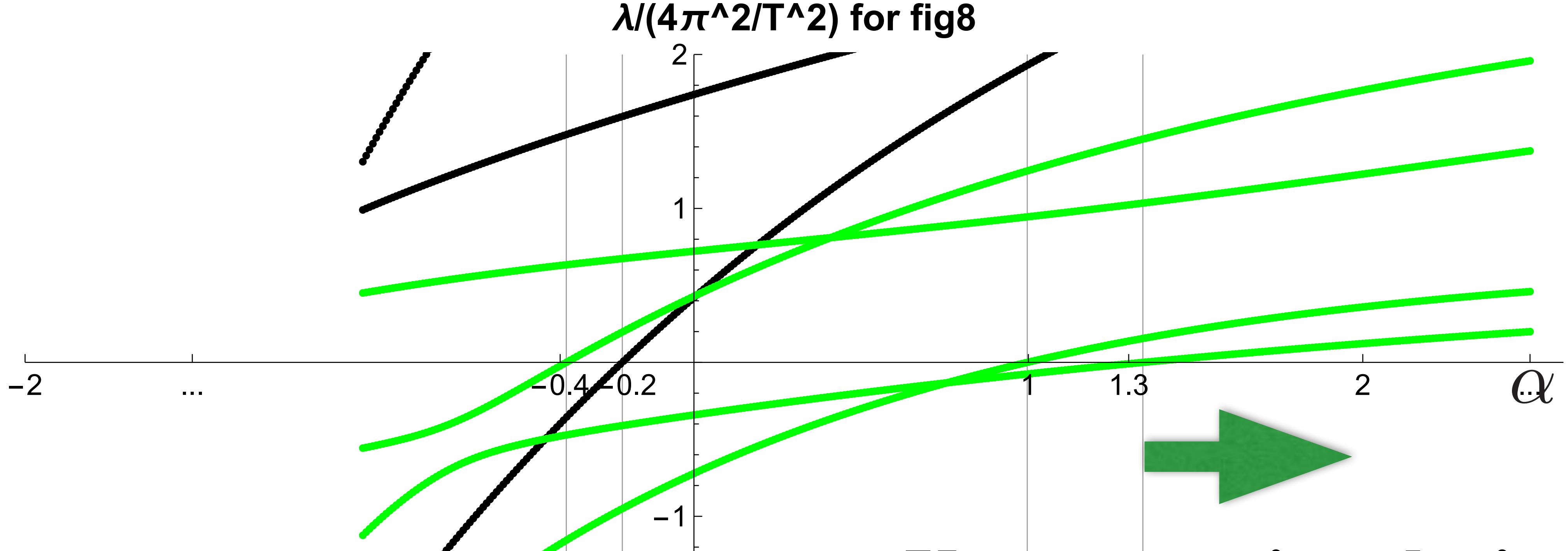


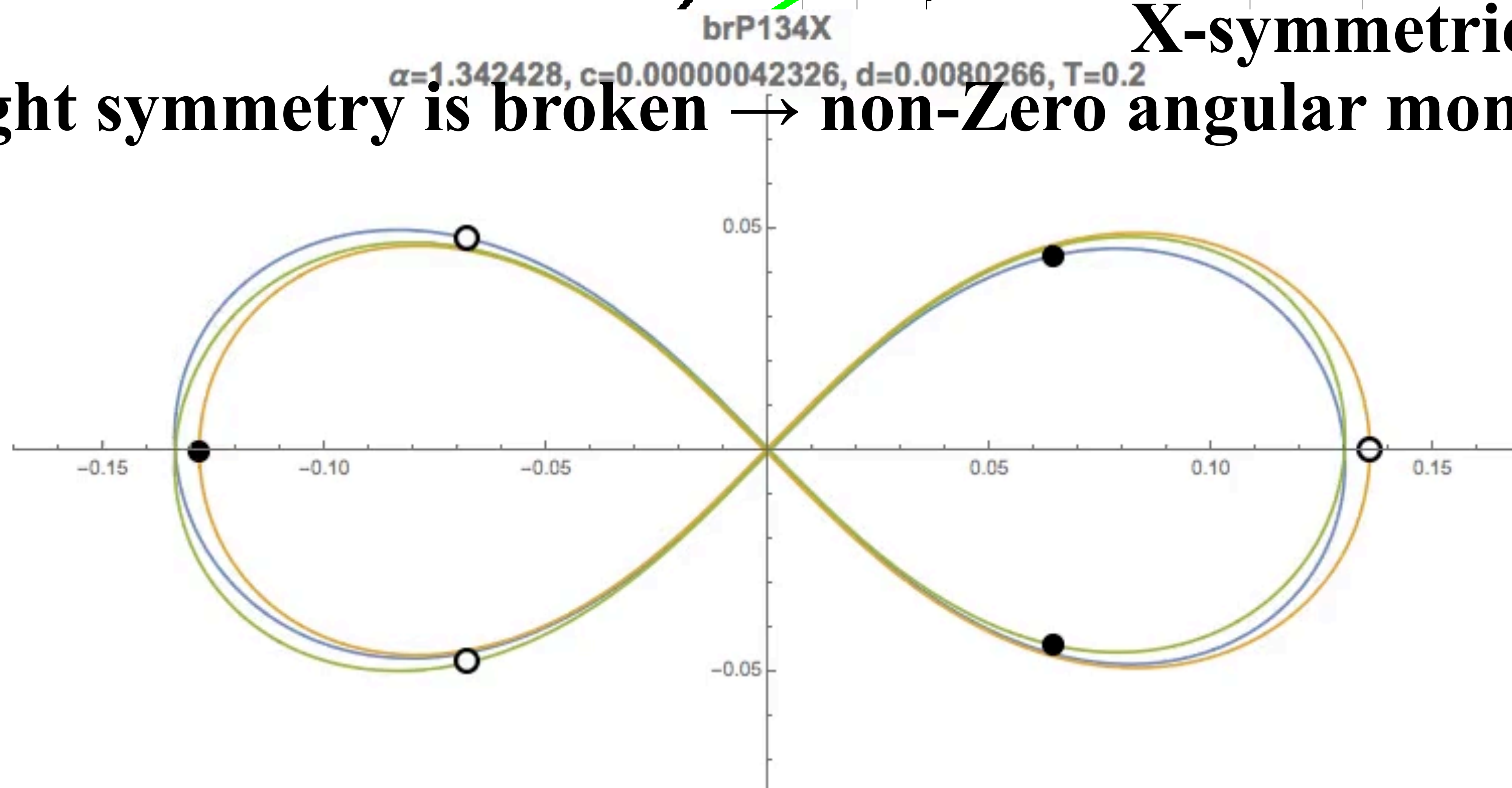
O-sym solution



bifurcations from figure-eight

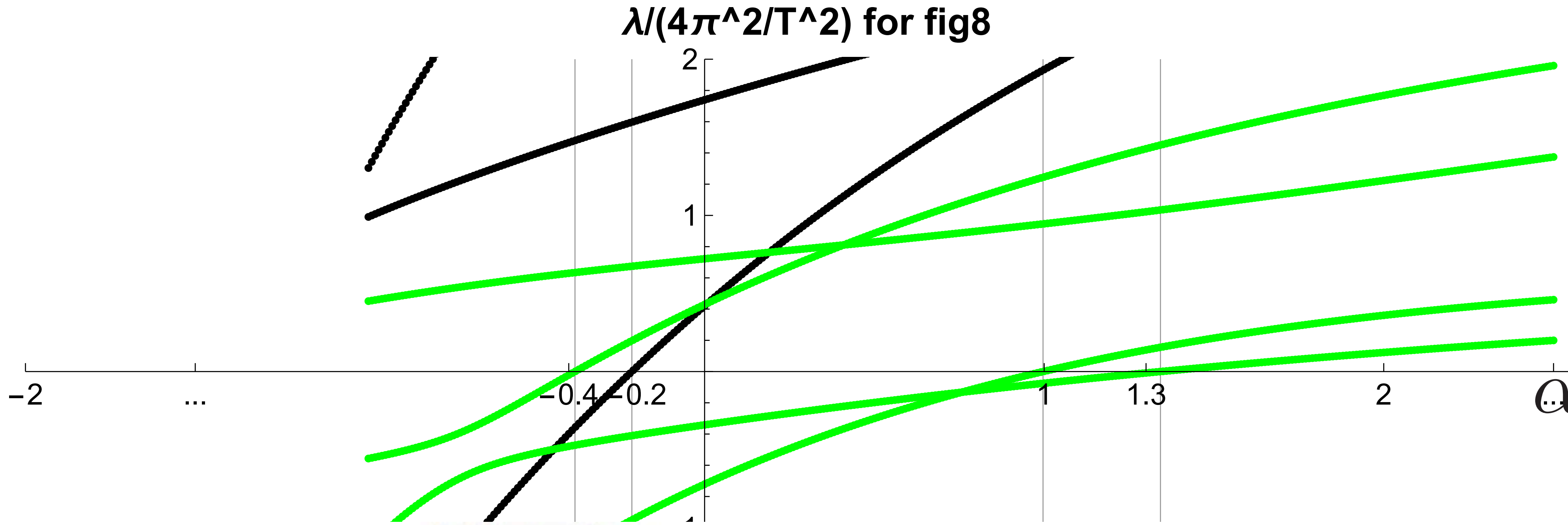


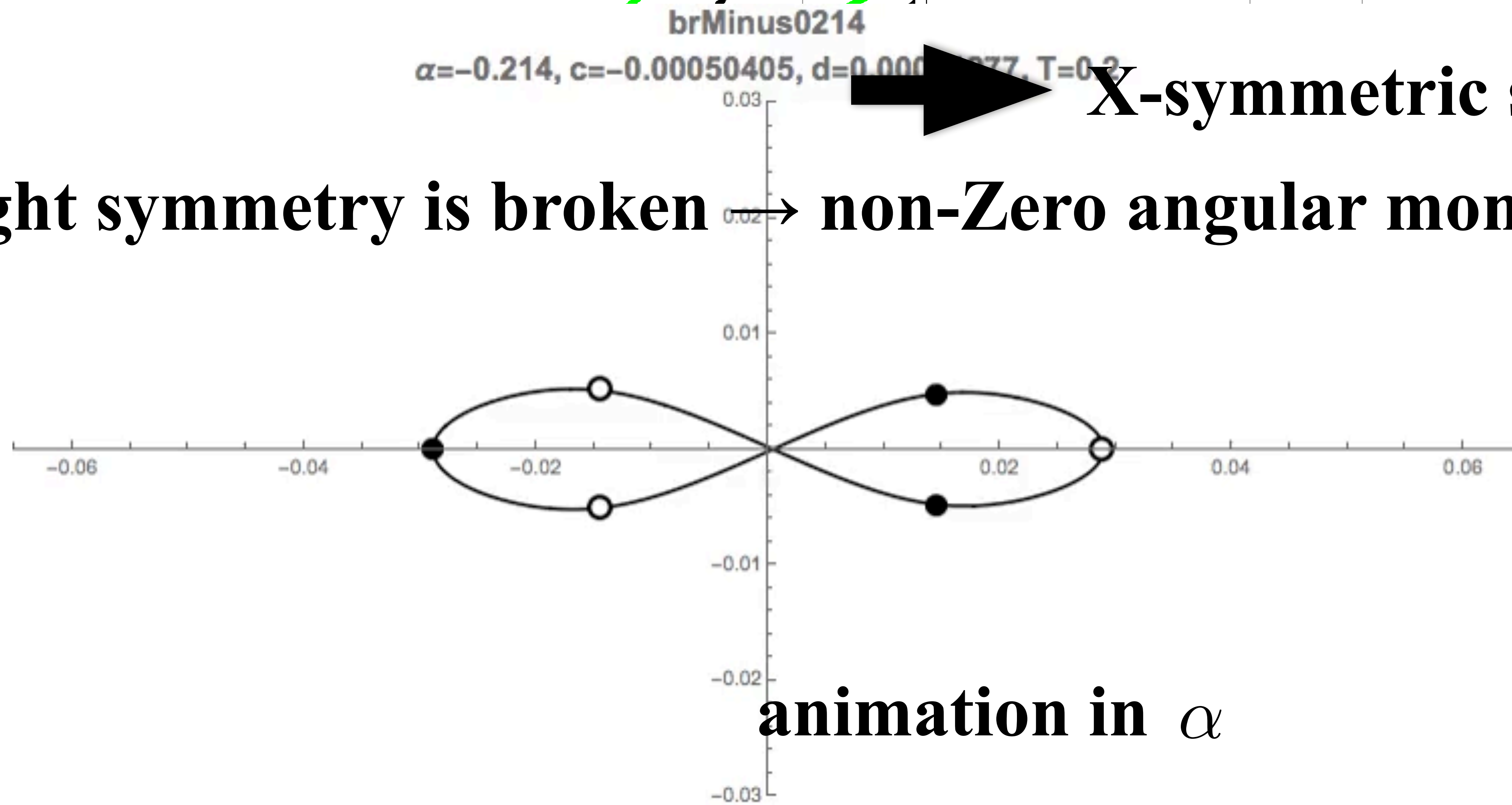
X-symmetric solution



left-right symmetry is broken → non-Zero angular momentum

bifurcations from figure-eight



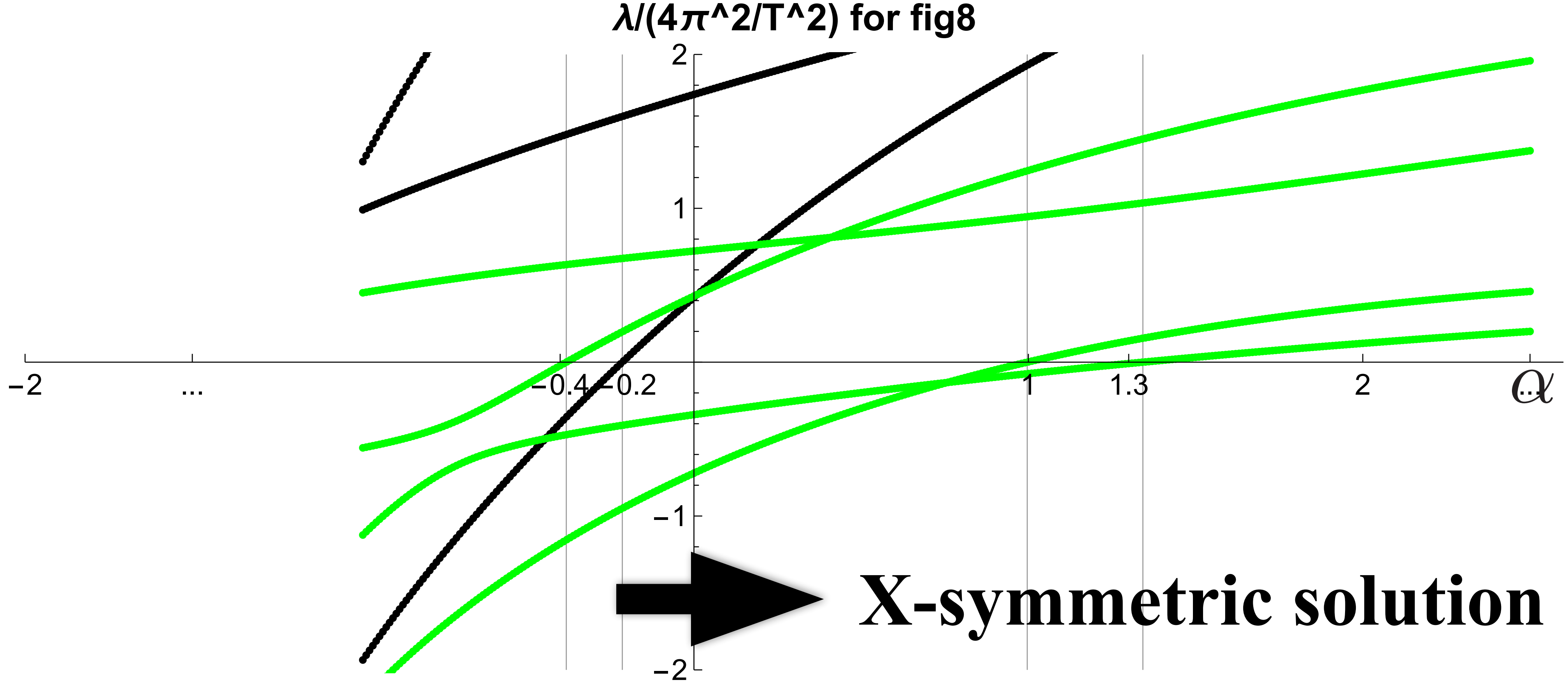


→ X-symmetric solution

left-right symmetry is broken → non-Zero angular momentum

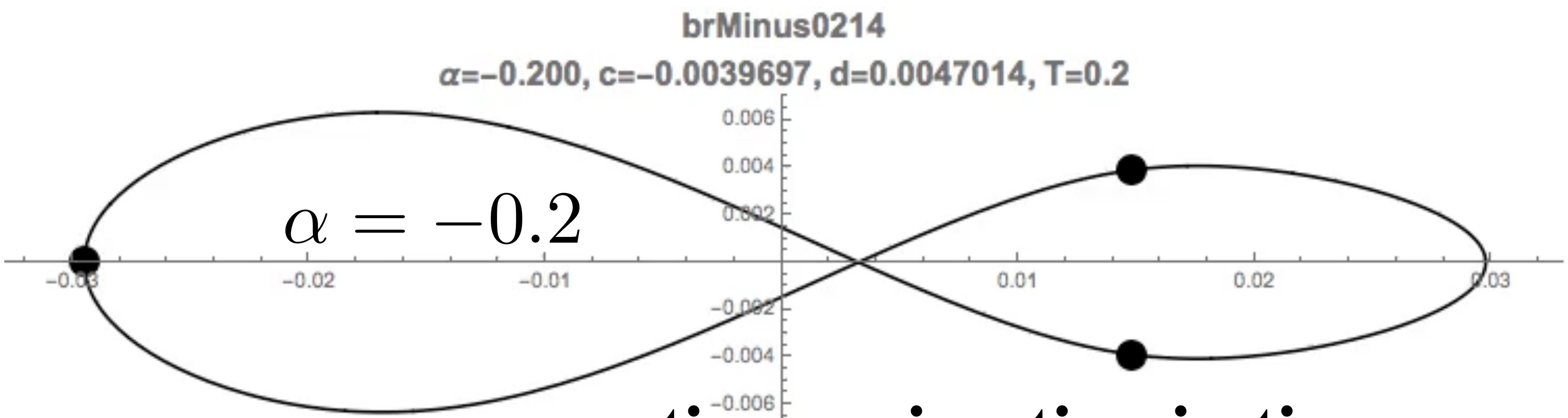
animation in α

bifurcations from figure-eight



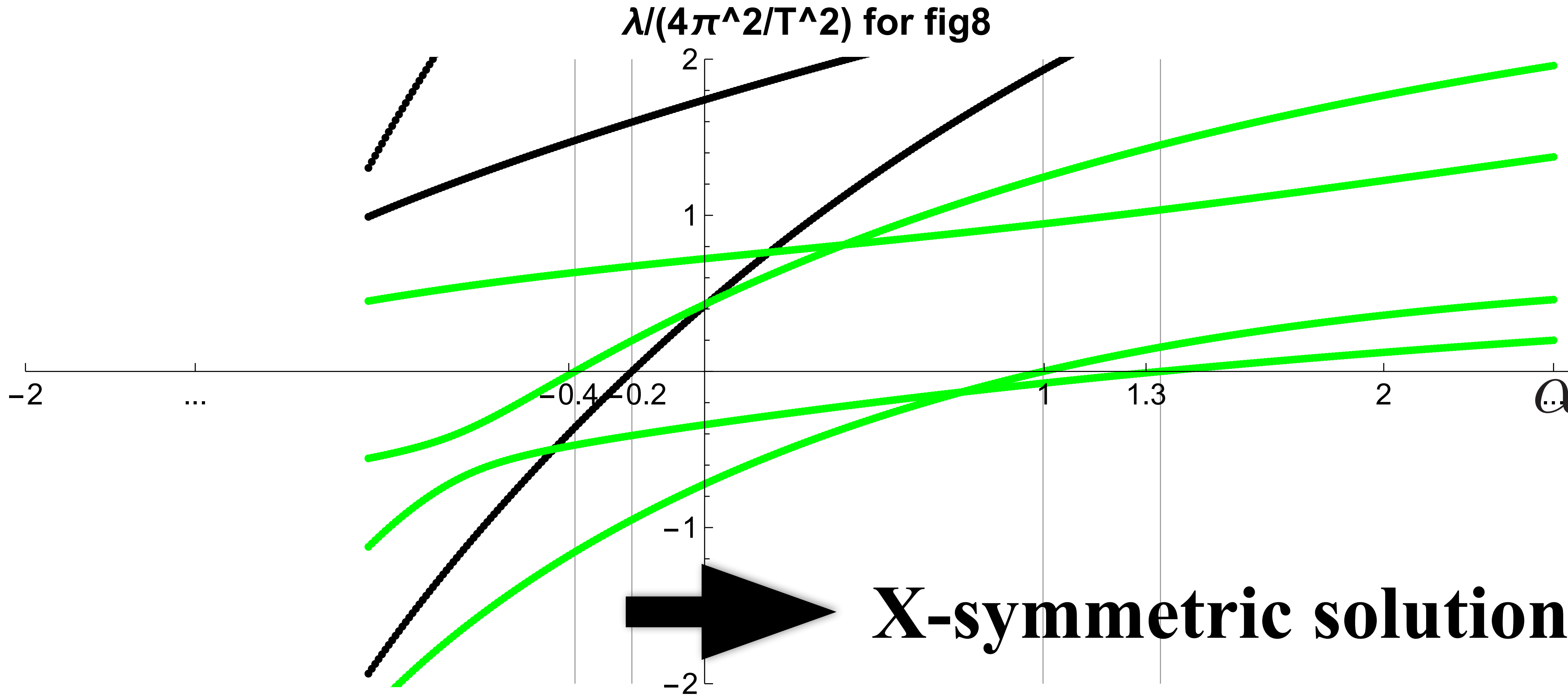
X-symmetric solution

left-right symmetry is broken → non-Zero angular momentum



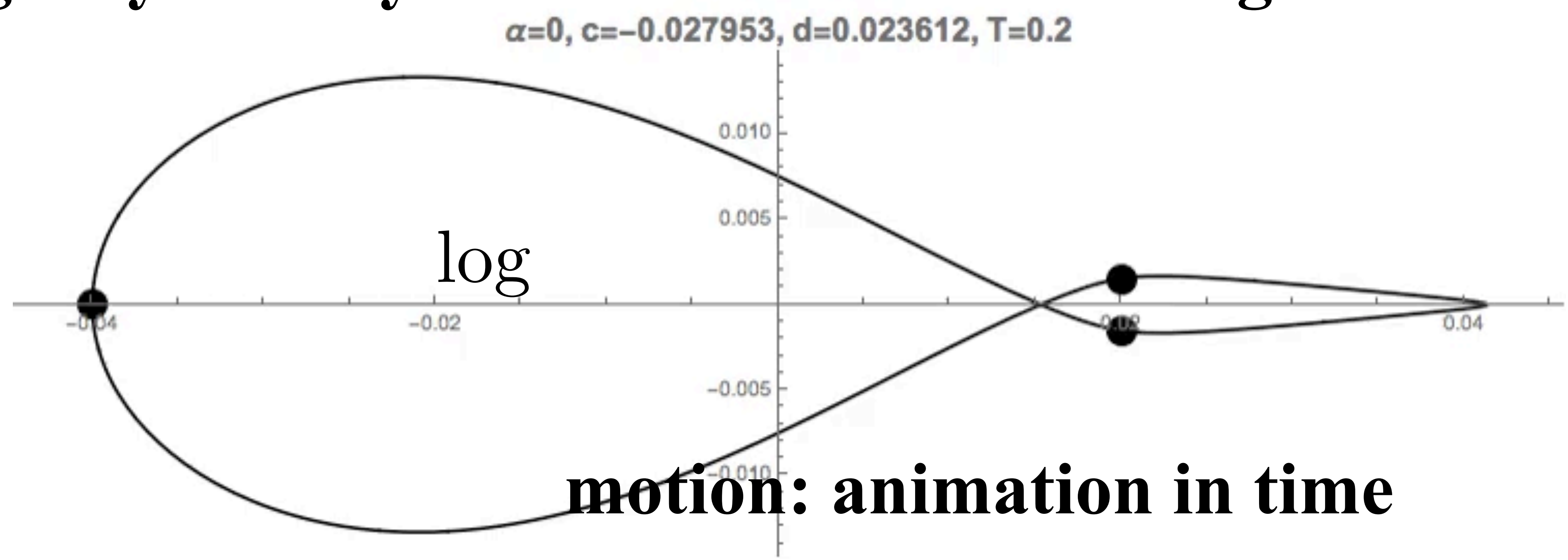
motion: animation in time

bifurcations from figure-eight



➡ X-symmetric solution

left-right symmetry is broken → non-Zero angular momentum



motion: animation in time

Linear stability, Floquet matrix

$$H = H(q,p) = H(x)$$
$$x = (q_0, q_1, q_2, p_0, p_1, p_2) \in \mathbb{R}^{12}$$
$$\frac{dx}{dt} = J\frac{\partial H}{\partial x},\ J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},\ E = 6 \times 6 \text{ identity}$$
$$x(t):\ \text{figure-eight},\ \delta x(t):\ \text{small variation}$$

equation of motion for small variation

$$\rightarrow \quad \frac{d}{dt}\delta x = J\frac{\partial^2 H}{\partial x^2}\delta x = B(t)\delta x$$
$$\frac{d}{dt}G(t) = B(t)G(t),\ G(0) = 12 \times 12 \text{ identity}$$

a solution for small variation

$$\rightarrow \quad \delta x(t) = G(t)\delta x(0)$$

Floquet matrix, eigenvalues

$$M = G(T),\ T = \text{period of figure-eight}$$

$$12 \text{ eigenvalues } \mu,\ M\phi = \mu\phi$$

μ: **characteristic multiplier**

properties: $\mu \to 1/\mu, \mu^*, 1/\mu^*$

one conserved quantity $\rightleftarrows$ **two** $\mu = 1$

centre of mass 2 $\rightleftarrows$ **4,**

angular momentum 1 $\rightleftarrows$ **2,** → eight $\mu = 1$ **"trivial"**

energy 1 $\rightleftarrows$ **2.**

$$12 - 8 = 4 : \text{ non trivial eigenvalues}$$

we concentrate on 4 non trivial eigenvalues
and neglect 8 "trivial" eigenvalues for a while

condition for bifurcation of the same period

the figure-eight: $q(t+T) = q(t)$

a bifurcated solution: $q(t+T) + \delta q(t+T) = q(t) + \delta q(t)$

$$\delta q(t+T) = \delta q(t)$$

$$\Downarrow$$

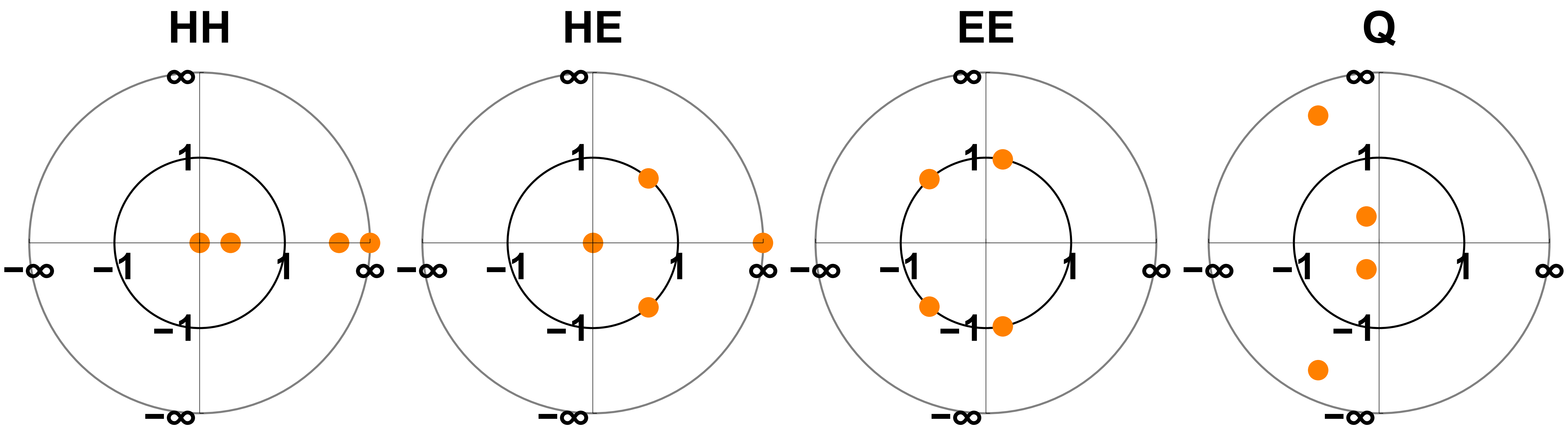
$$\delta q(T) = G(T)\delta q(0) = \delta q(0)$$

$\delta q(0)$ must be an eigenvector of $M = G(T)$
with eigenvalue $\mu = 1$

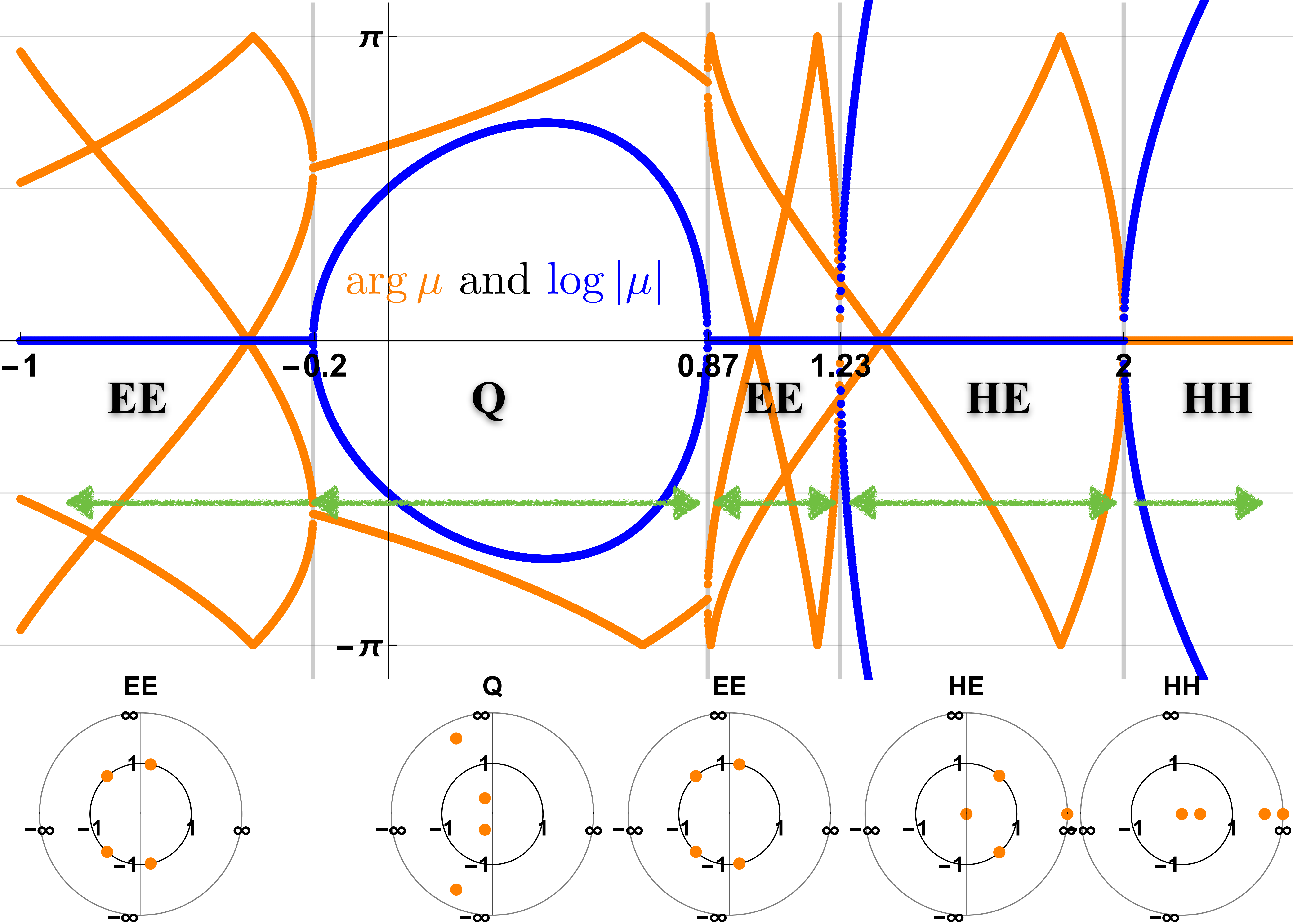
a necessary condition

four possible combinations of 4 non-trivial eigenvalues

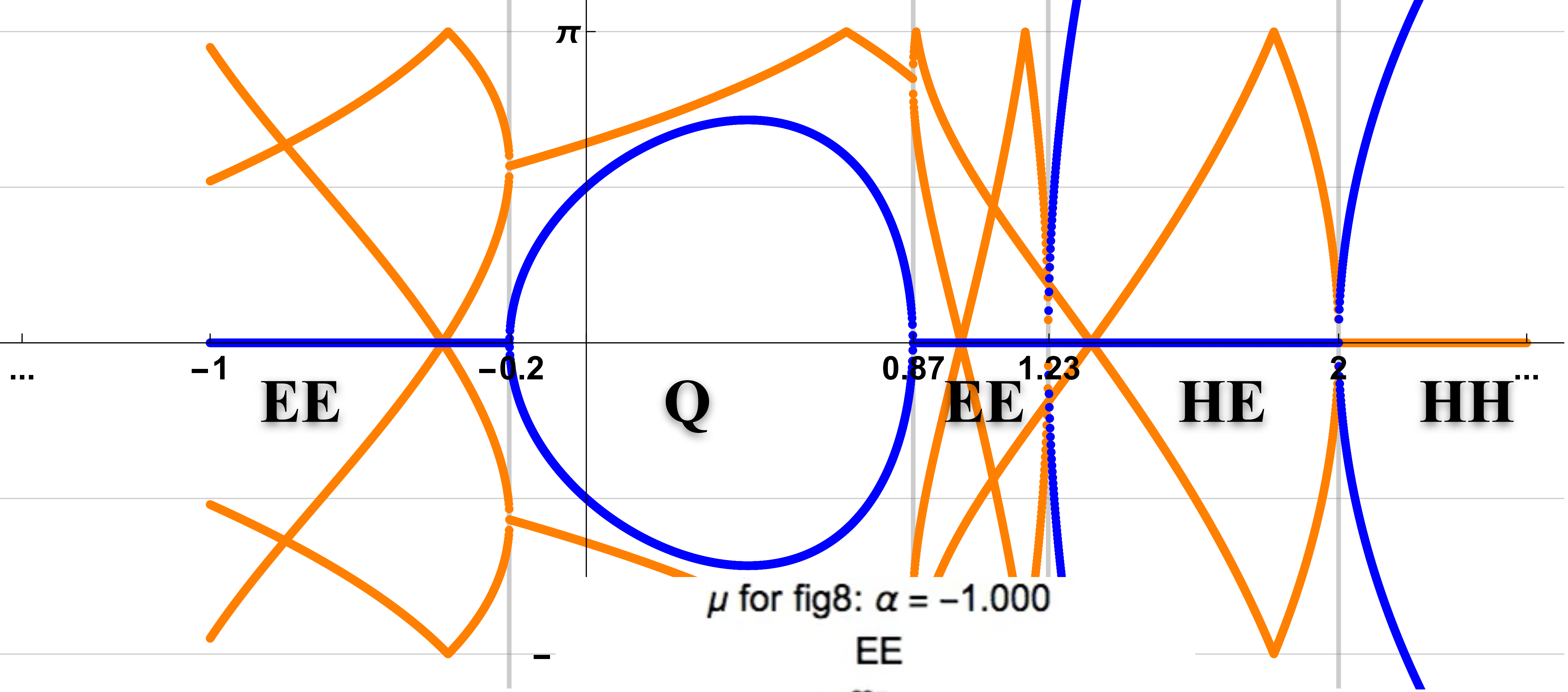
$$\mu \to \mu^{-1}, \mu^*, (\mu^*)^{-1}$$

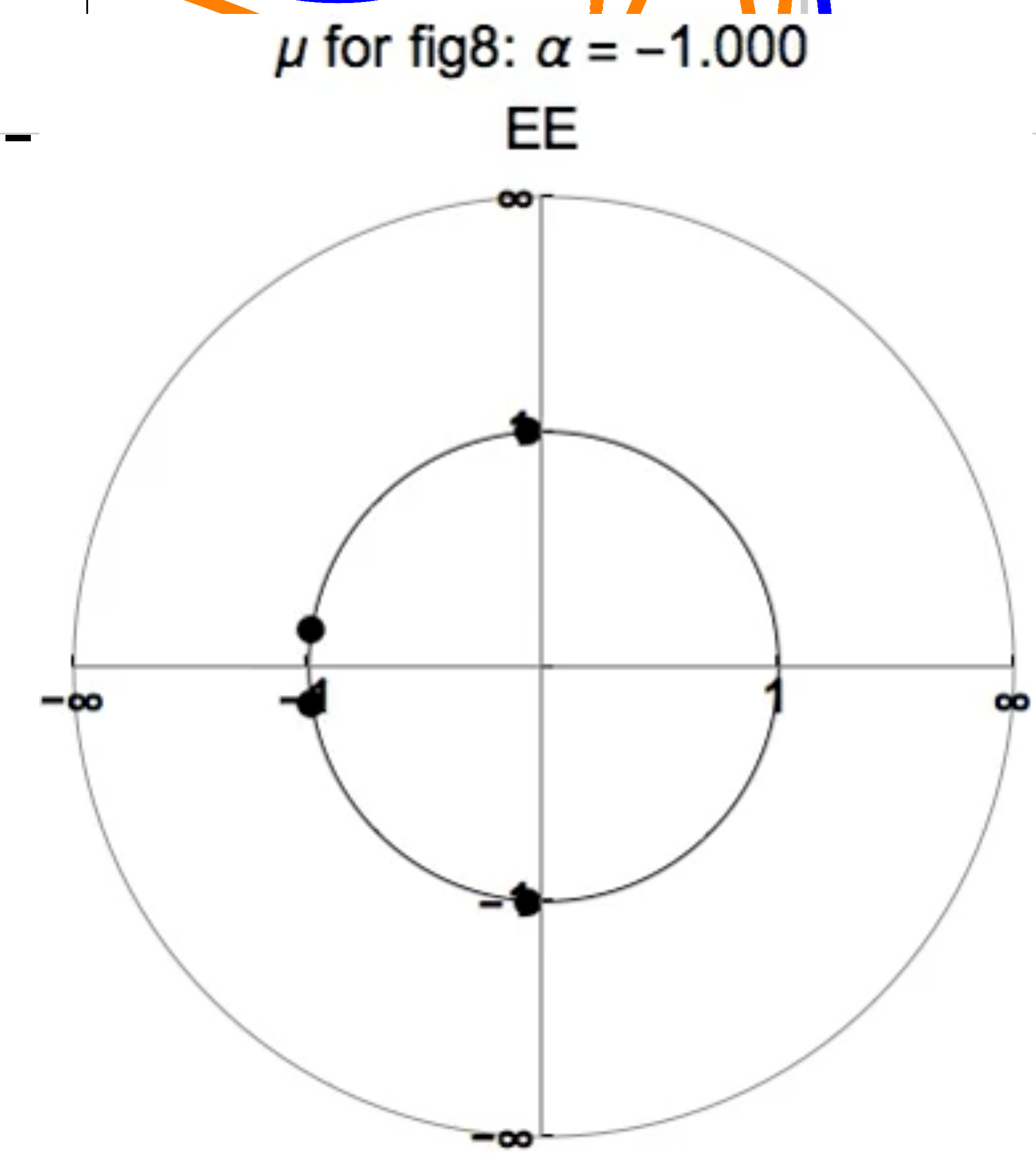


Hyperbolic, **E**lliptic, **Q**uartet

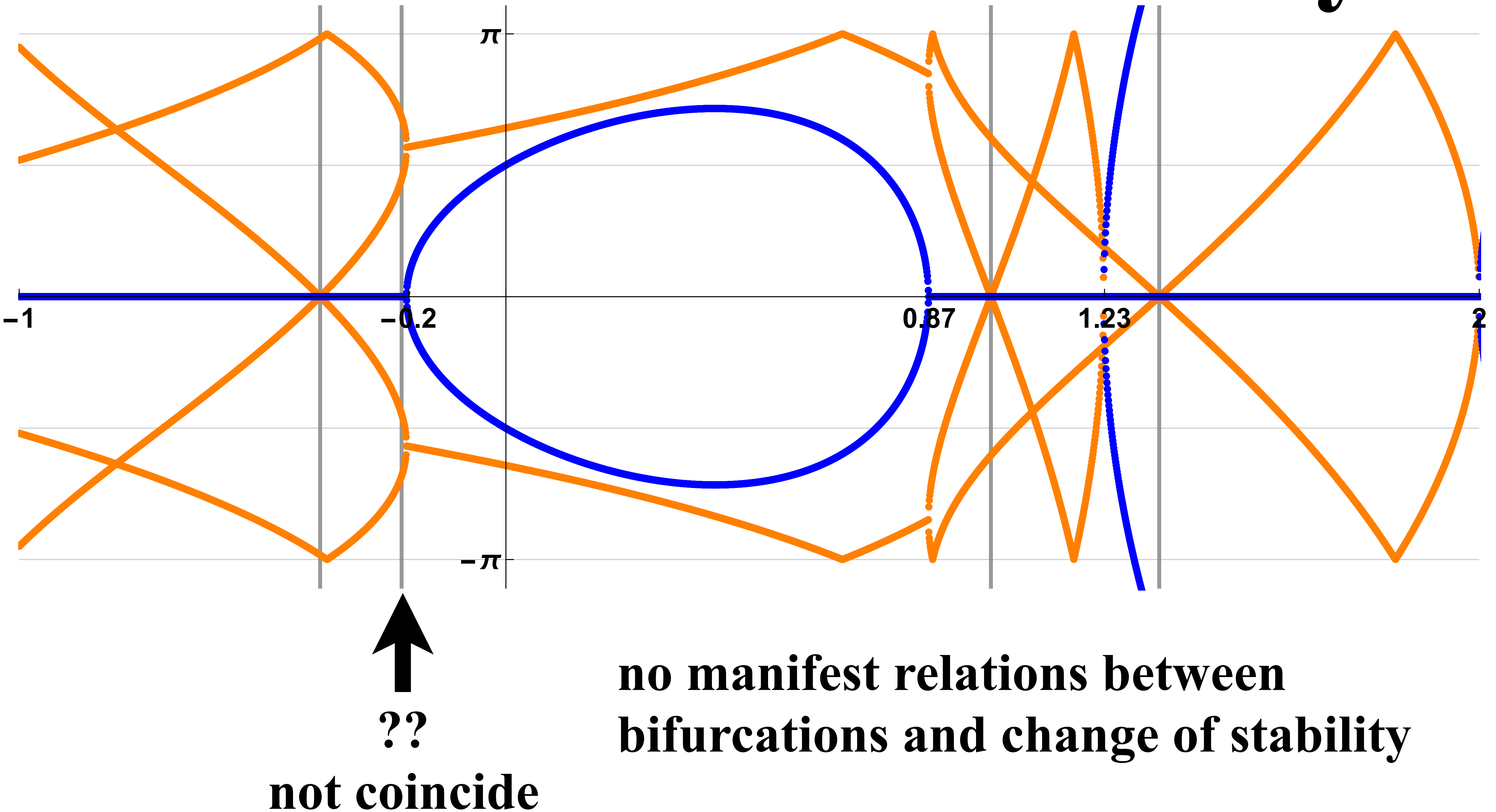


π
−π
arg μ and log |μ|
−1
−0.2
0.87
1.23
2
EE
Q
EE
HE
HH
EE
Q
EE
HE
HH


π
...
−1
−0.2
0.87
1.23
2
...
EE
Q
EE
HE
HH
−


μ for fig8: α = −1.000
EE
∞
−∞
−1
1
∞
−∞

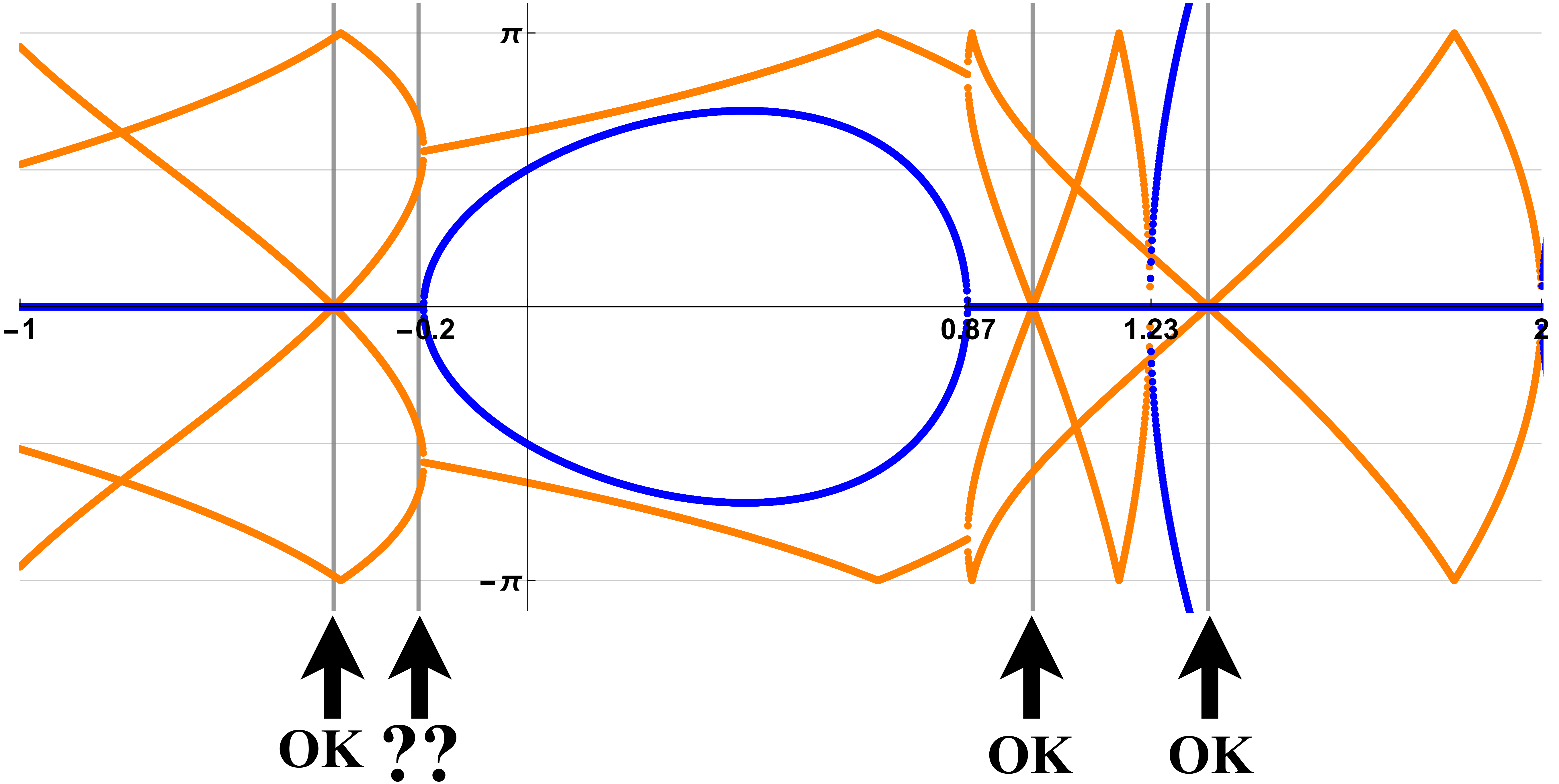
bifurcations and stability



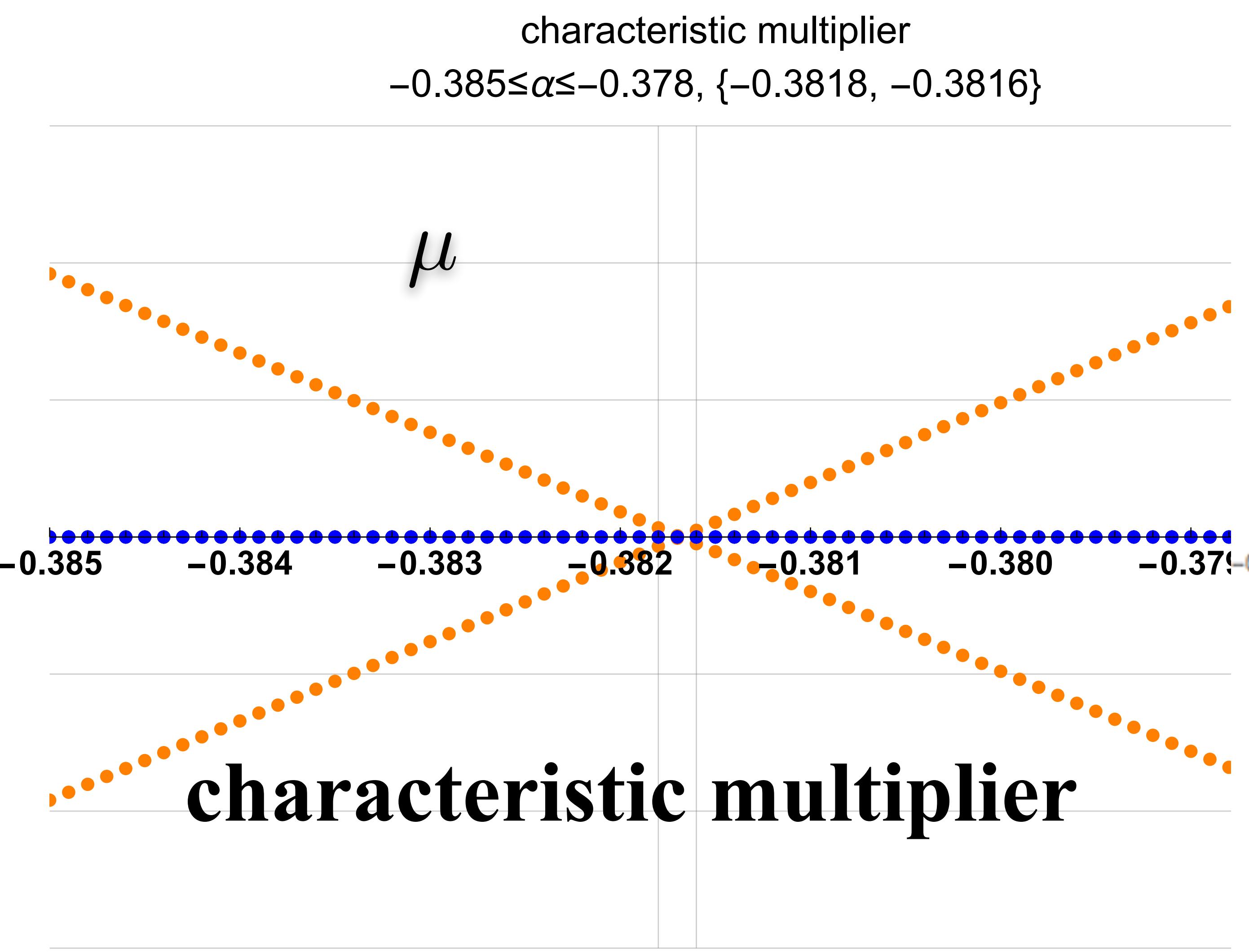
no manifest relations between bifurcations and change of stability

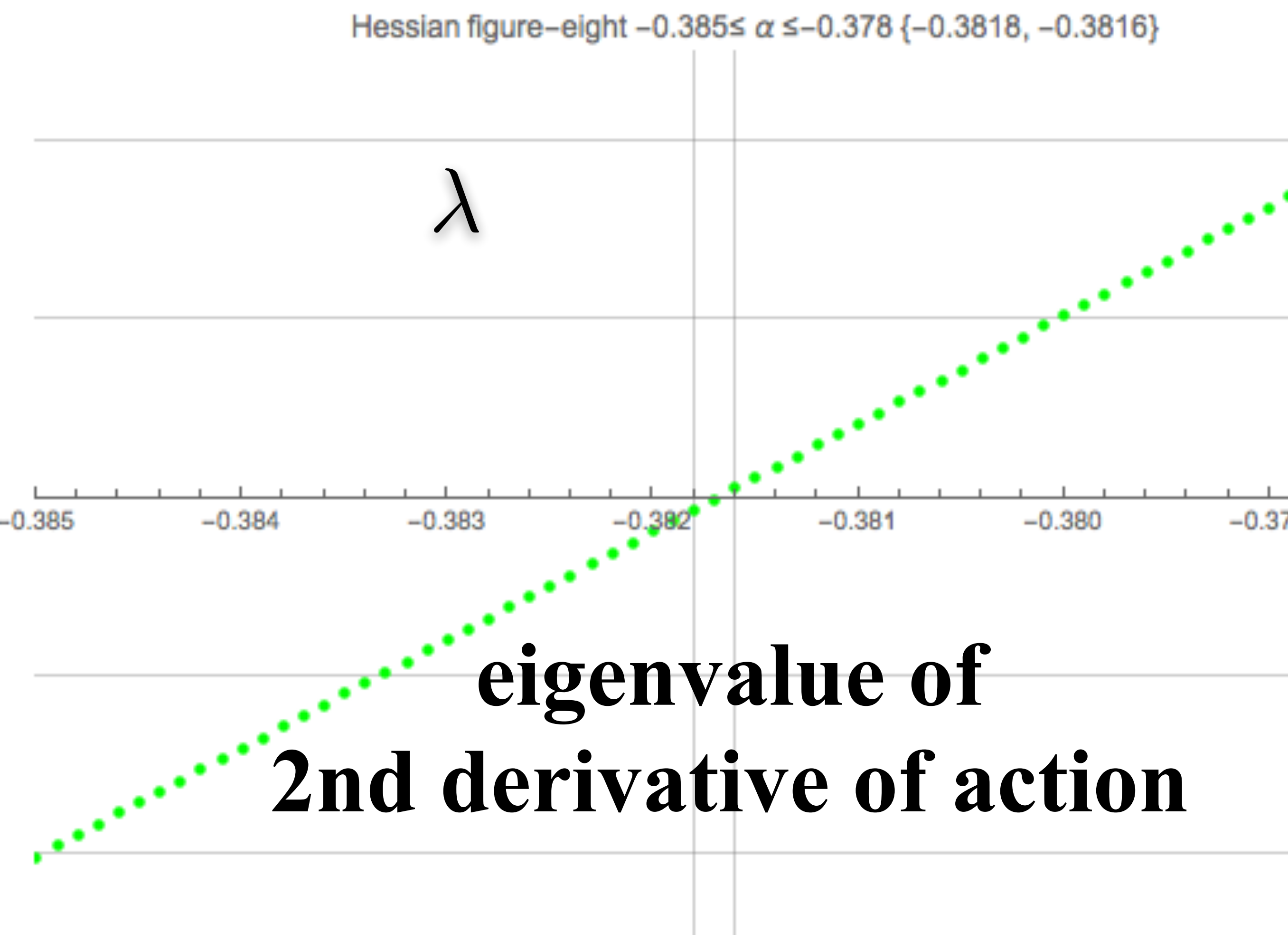
Eigenvalues of Floquet matrix and Bifurcations

$$\mu = 1 \Leftrightarrow \arg \mu = 0, \;\; \log |\mu| = 0$$



OK for 3 cases, but ?? for the rest one

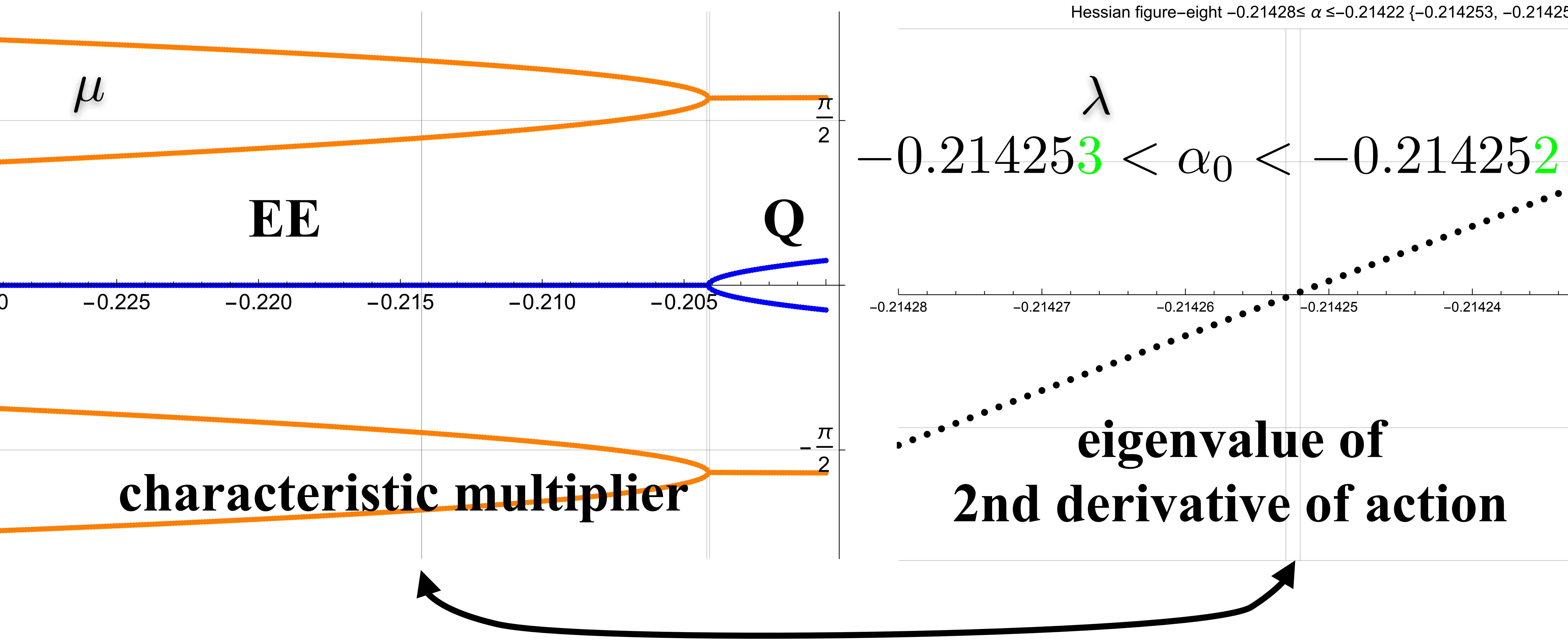


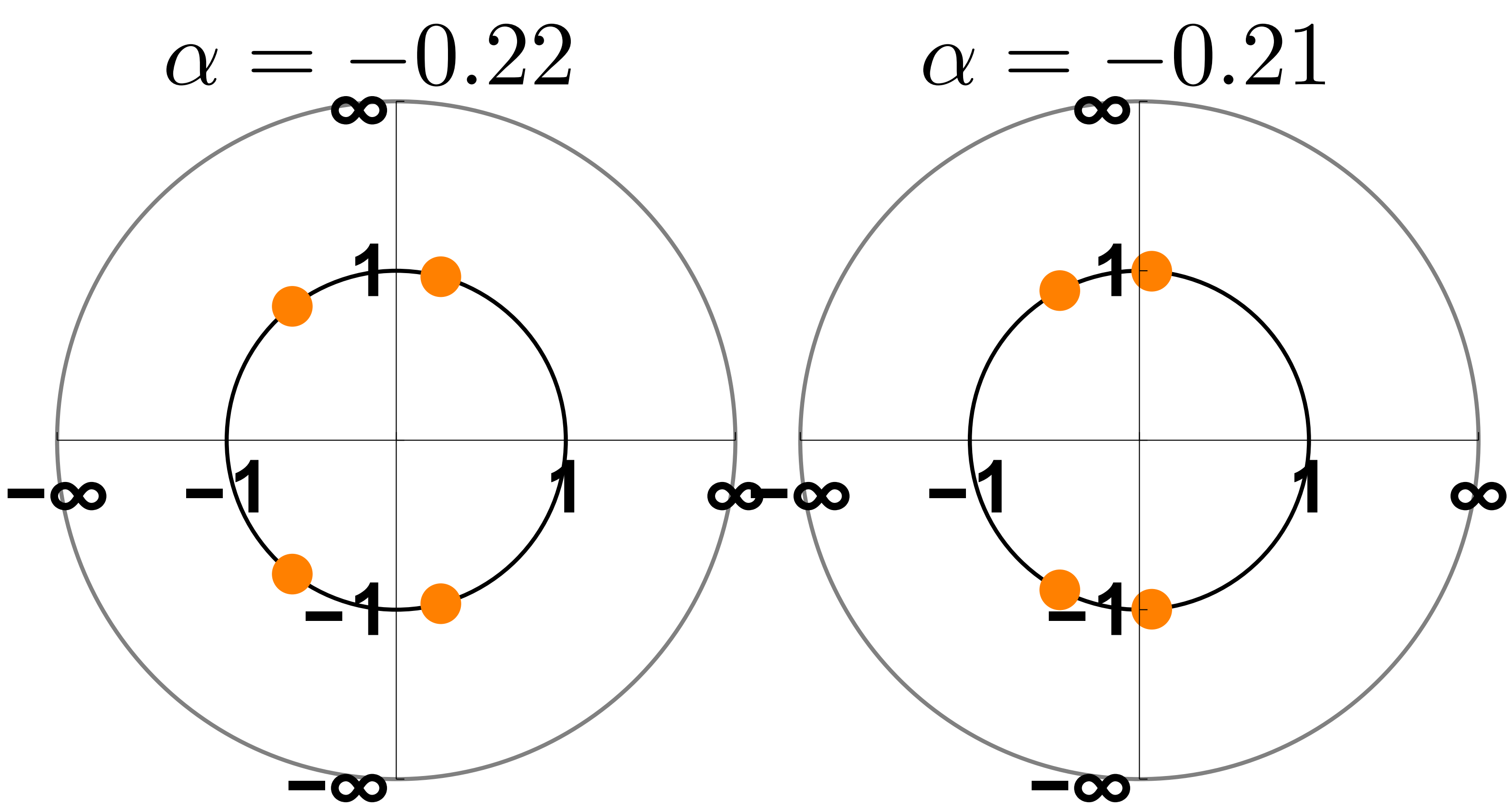


$\mu = 1$ and $\lambda = 0$

both double lines: $-0.3818 < \alpha_0 < -0.3816$

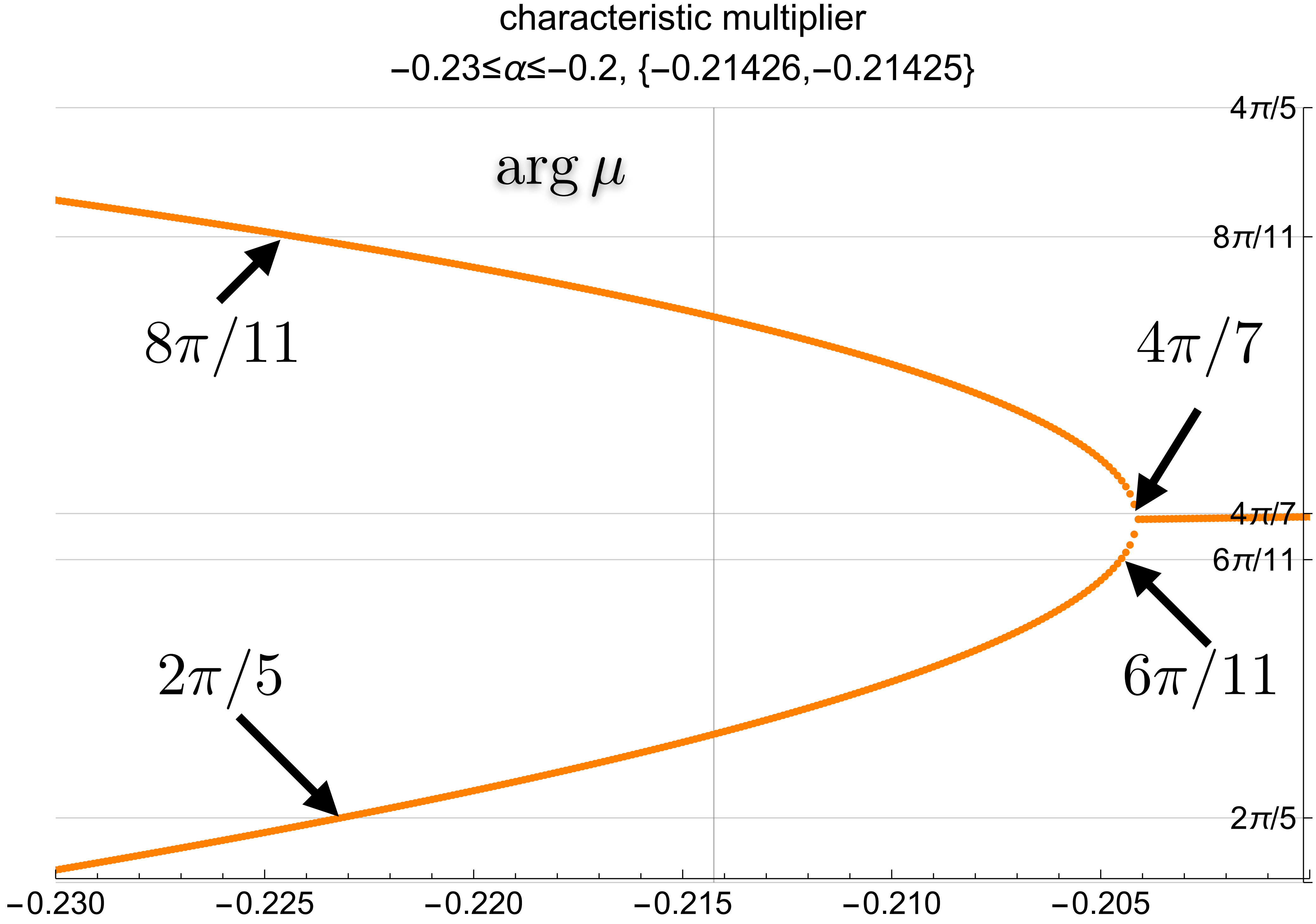
good agreement

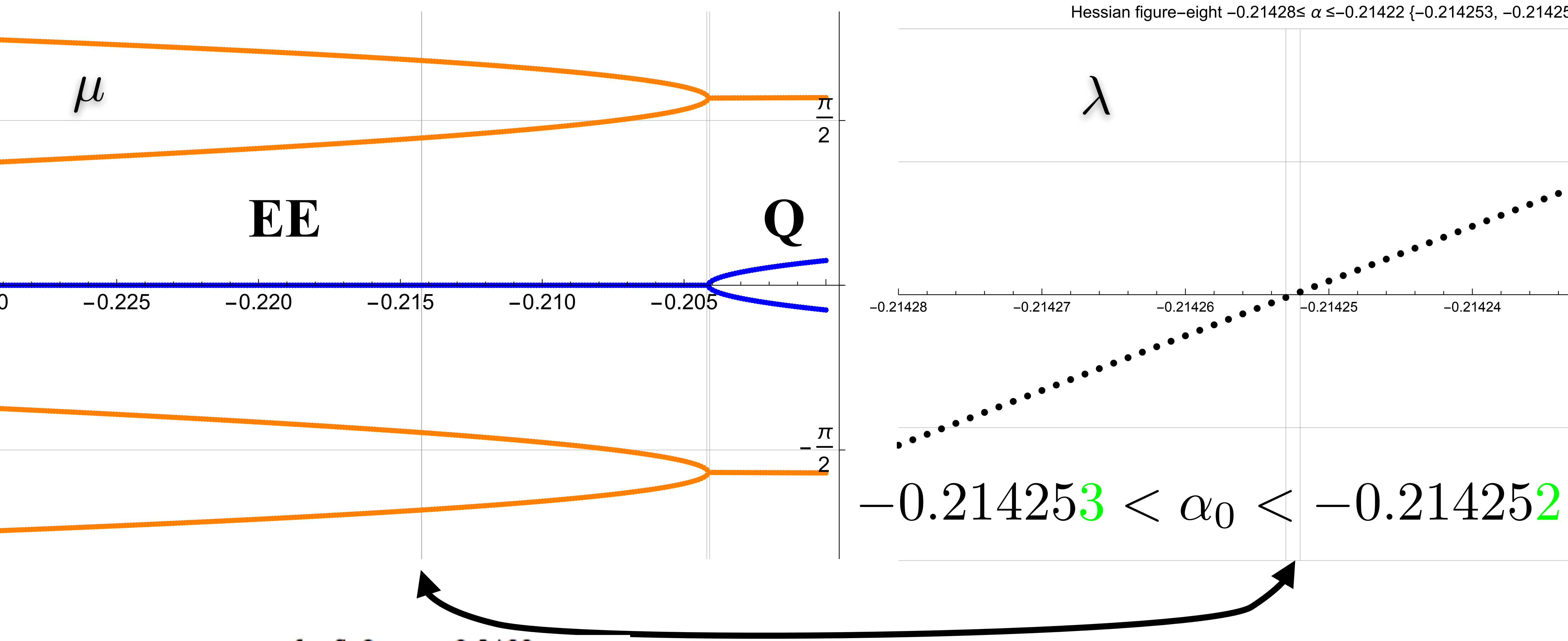


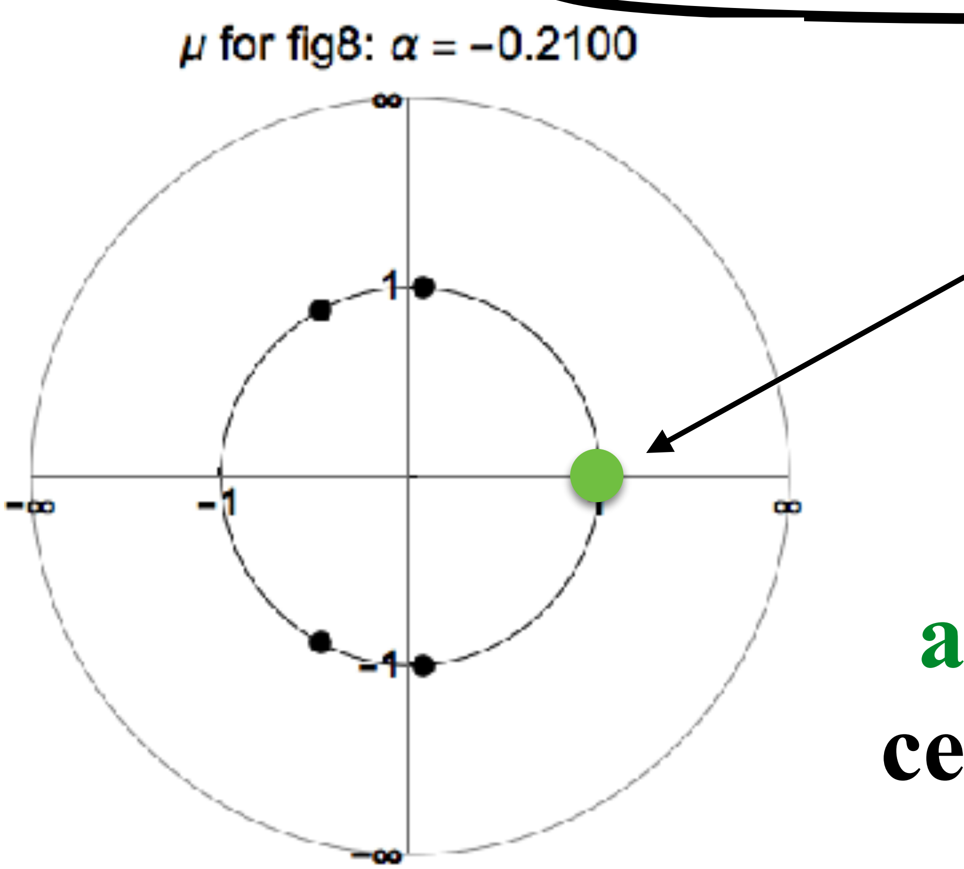


the bifurcation pointis in Elliptic-Elliptic region

error in numerical calculations? four check points







here!
we have 8 "trivial"

$$\mu = 1$$

angular momentum,
centre of mass, energy

どうすればよいか？

$$\delta q(t) = G(t)\delta q(0)$$

$M = G(T)$ の情報しか用いていない

$G(t),\ 0 \le t$ の情報を使えば？

変分方程式 $\dfrac{d}{dt}\delta x = J\dfrac{\partial^2 H}{\partial x^2}\delta x = B(t)\delta x$

の高階微分

まとめ

- 作用の2階微分の固有値問題と分岐
  - ☆1:1に対応
  - ☆固有関数から分岐解の対称性，解の個数
- Floquet行列の固有値問題と分岐
  - ☆安定性と分岐：直接的な関係は見えない（今回の範囲では）
    - ・同周期の分岐点で安定性は変化していない
    - ・安定性が変化する点で同周期の分岐は起きていない
  - ☆固有値と分岐：“non-trivial”な固有値だけでは，見えない分岐も
    - ・“trivial”な固有値（おそらく角運動量）も考慮する必要
    - ・ $M = G(T)$ のみではなく $G(t)$ の情報を
    - ・変分方程式の高階微分
- 分岐の詳細はどちらも高次の微係数が必要