Motion in shape for planar three-body problem

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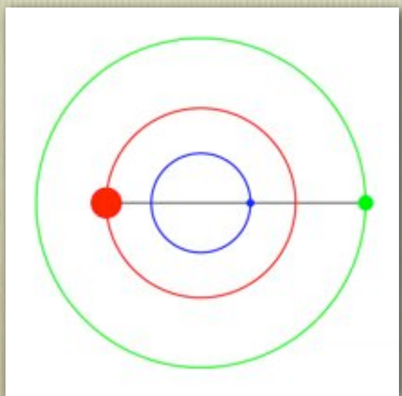
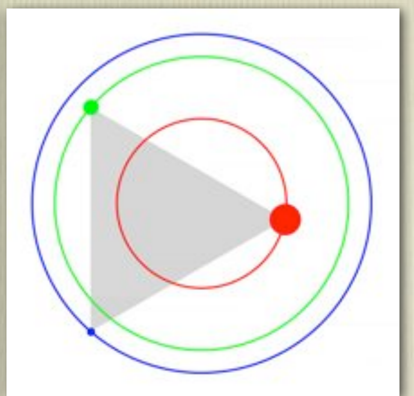
Saari's conjecture

In N-body problem,
if $I = \sum m_k |q_k|^2 =$ constant,
the motion is relative equilibrium. (1970)

Saari's conjecture

for 3-body:
Euler-Lagrange circle solution
$\Leftrightarrow I = \sum m_k |q_k|^2 =$ constant.





35 years after, Moeckel proved ...

A Proof of Saari's Conjecture for the Three-Body Problem in R^d (May 20, 2005)

April 7, 2005 at Saarifest 2005, Guanajuato, Mexico

The next day, Saari extended his conjecture

If configurational measure $\mu = I^{\alpha/2} U =$ constant, then the motion is homographic.

$$I = \sum_{k=1,2,\ldots,N} m_k |q_k|^2,$$

$$U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|q_i - q_j|^\alpha}$$

$$\mu(\lambda q_k) = \mu(q_k), \quad q_k, \lambda \in \mathbb{C}.$$

April 8, 2005 at Saarifest 2005, Guanajuato, Mexico

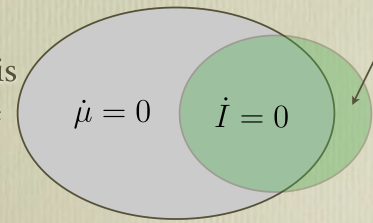
Saari and Extended Saari

Lagrange-Jacobi $\ddot{I} = 4E + \frac{2(2-\alpha)}{\alpha} U$

for $\alpha \neq 2$
$\dot{I} = 0 \Rightarrow \dot{U} = 0 \Rightarrow \mu = I^\alpha U =$ constant.

for $\alpha = 2$
$\dot{I} = 0$ doesn't mean $\dot{U} = 0$ or $\dot{\mu} = 0$.
there are counter examples of the original Saari's conjecture

extended Saari is expected to be true for any α



$\alpha = 2$
there are counter examples for original Saari

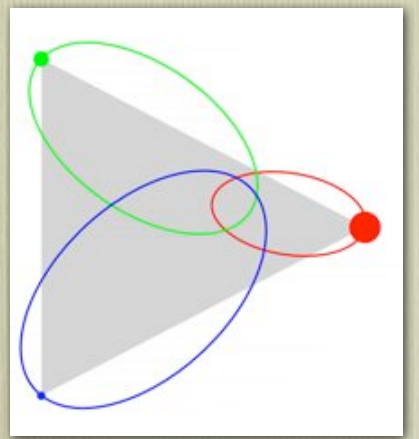
Saari's extended conjecture

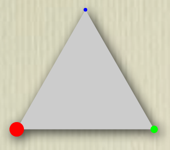
If configurational measure $\mu = I^{\alpha/2} U =$ constant, then the motion is homographic.

Hereafter, we call this conjecture
"Saari's homographic conjecture"
or simply
"Saari's conjecture".

now, Saari's conjecture contains ...

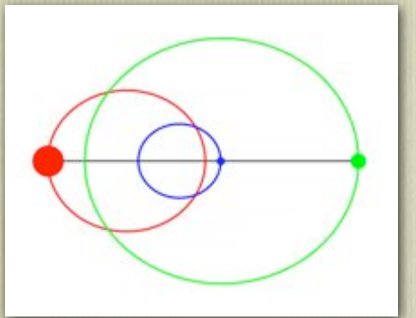
Lagrange solution

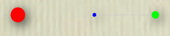




the shape is unchanged

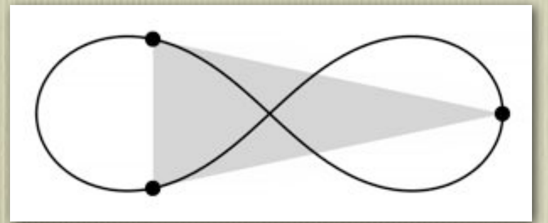
Euler solution

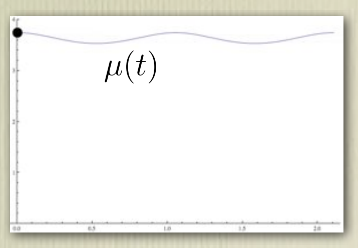




the shape is unchanged

figure-eight solution Newton potential





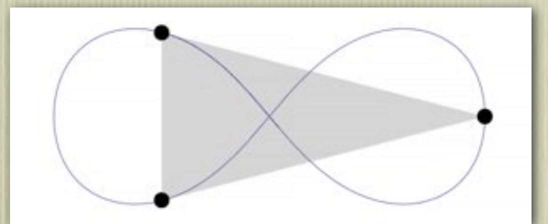
$q_k(t)$

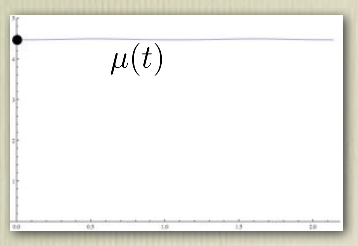
$$\frac{5}{\sqrt{2}} = 3.535... \leq \mu(t) \leq 3.745...$$

$$\mu = \sqrt{\sum |q_k|^2} \left(\sum \frac{1}{|q_i - q_j|} \right)$$

$$\frac{\Delta\mu}{\mu} = 0.059...$$

figure-eight solution under 1/r^2 potential





$q_k(t)$

$$4.474... \leq \mu \leq \frac{9}{2} = 4.5$$

$$\mu = \left(\sum |q_k|^2 \right) \left(\sum \frac{1}{|q_i - q_j|^2} \right)$$

$$\frac{\Delta\mu}{\mu} = 0.0057...$$

Saari's conjecture

⇩

pay attention to Shape variable

- what is the Shape variable ?
- equation of motion for the Shape variable ?

Degree of freedom

$q_1, q_2, q_3 \in \mathbb{C} \Rightarrow 6$

center of mass $\Rightarrow 2$
size $\Rightarrow 1$
rotation $\Rightarrow 1$

$\therefore$ shape $\Rightarrow 2$

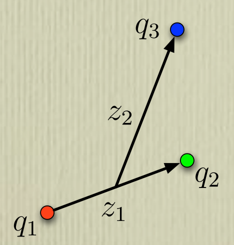
Shape variable

January 18, 2011, I received a mail from Richard Montgomery with an unpublished preprint written in 2007.

In the preprint, Moeckel and Montgomery ...

- the Shape variable for Planar 3-body,
- the Lagrangian,
- the equations of motion.

The Shape variable



Jacobi variables:

$$z_1 = q_2 - q_1, \ z_2 = q_3 - \frac{q_1 + q_2}{2} = \frac{3}{2} q_3.$$

Shape variable: $\zeta = \frac{z_2}{z_1} = \frac{3}{2} \frac{q_3}{q_2 - q_1}.$

Moeckel & Montgomery 2007

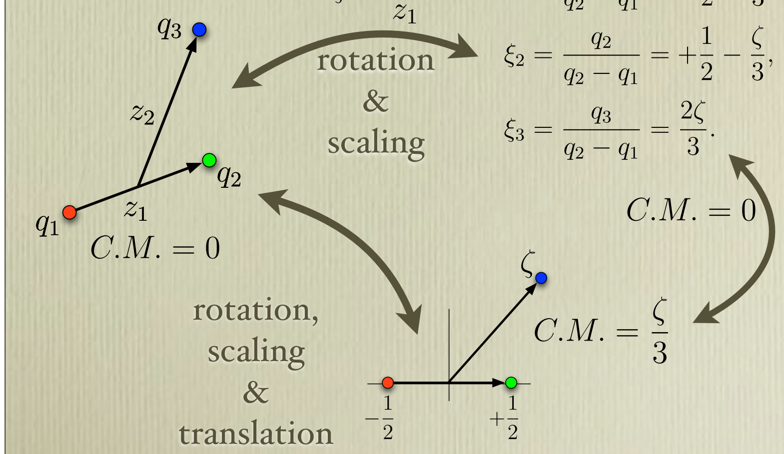
$m_k = 1$ for simplicity
$q_1 + q_2 + q_3 = 0$

$$\Rightarrow \quad \xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_2 = \frac{q_2}{q_2 - q_1} = +\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_3 = \frac{q_3}{q_2 - q_1} = \frac{2\zeta}{3}.$$

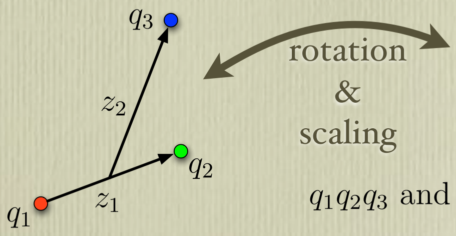
geometric interpretation

$$\zeta = \frac{z_2}{z_1}$$

$$\xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_2 = \frac{q_2}{q_2 - q_1} = +\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_3 = \frac{q_3}{q_2 - q_1} = \frac{2\zeta}{3}.$$



The Shape variable



$$\xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{1}{2} - \frac{\zeta}{3},$$

$$\xi_2 = \frac{q_2}{q_2 - q_1} = +\frac{1}{2} - \frac{\zeta}{3},$$

$$\xi_3 = \frac{q_3}{q_2 - q_1} = \frac{2\zeta}{3}.$$

$q_1 q_2 q_3$ and $\xi_1 \xi_2 \xi_3$ are similar

$$\therefore q_k = r e^{i\theta} \frac{\xi_k}{\sqrt{\sum |\xi_\ell|^2}}, \qquad r \geq 0, \; \theta \in \mathbb{R}.$$

r, θ, ζ: generalized coordinates

Lagrangian

$$q_k = r e^{i\theta} \frac{\xi_k}{\sqrt{\sum |\xi_\ell|^2}} \quad \Rightarrow \quad L = \frac{1}{2} K + \frac{1}{\alpha} U$$

$$\frac{K}{2} = \frac{1}{2} \sum |\dot{q}_k|^2 = \frac{\dot{r}^2}{2} + \frac{r^2}{2} \left(\dot{\theta} + \frac{\frac{2}{3} \zeta \wedge \dot{\zeta}}{\frac{1}{2} + \frac{2}{3} |\zeta|^2} \right)^2 + \frac{r^2}{6} \frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2},$$

$$U = \sum \frac{1}{|q_i - q_j|^\alpha} = \frac{\mu(\zeta)}{r^\alpha},$$

$$\mu = \left(\frac{1}{2} + \frac{2}{3} |\zeta|^2 \right)^{\alpha/2} \left(1 + \frac{1}{|\zeta - \frac{1}{2}|^\alpha} + \frac{1}{|\zeta + \frac{1}{2}|^\alpha} \right).$$

Angular momentum

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow C = \frac{\partial L}{\partial \dot{\theta}} = r^2 \left(\dot{\theta} + \frac{\frac{2}{3} \zeta \wedge \dot{\zeta}}{\frac{1}{2} + \frac{2}{3} |\zeta|^2} \right) = \text{constant}$$

Decomposition of Kinetic energy

$$\frac{K}{2} = \frac{\dot{r}^2}{2} + \frac{C^2}{2r^2} + \frac{r^2}{6} \frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2},$$

kinetic energy for size + rotation + shape

Size

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad \Rightarrow \quad \ddot{r} = \frac{C^2}{r^3} + \frac{r}{3} \frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2} - \frac{\mu(\zeta)}{r^{\alpha+1}}$$

by a few lines calculations, we get

$$\frac{d}{dt} \left(\frac{r^4}{6} \frac{|\dot{\zeta}|^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)^2} \right) = \frac{r^{2-\alpha}}{\alpha} \frac{d\mu}{dt} \qquad \text{: Saar's relation}$$

r^2 times
kinetic energy for the shape motion

$$\Rightarrow \frac{d}{ds} \left(\frac{1}{6} \left| \frac{d\zeta}{ds} \right|^2 \right) = \frac{r^{2-\alpha}}{\alpha} \frac{d\mu}{ds}, \qquad \frac{r^2}{\left(\frac{1}{2} + \frac{2}{3}|\zeta|^2\right)} \frac{d}{dt} = \frac{d}{ds}$$

Saari's relation

$$\frac{d}{dt}\left(\frac{r^4}{6}\frac{|\dot\zeta|^2}{\left(\frac{1}{2}+\frac{2}{3}|\zeta|^2\right)^2}\right)=\frac{r^{2-\alpha}}{\alpha}\frac{d\mu}{dt}$$

comes from the structure of the Kinetic energy

$$\frac{K}{2}=\frac{1}{2}\sum|\dot q_k|^2=\frac{\dot r^2}{2}+\frac{r^2}{2}\left(\dot\theta+\frac{\frac{2}{3}\zeta\wedge\dot\zeta}{\frac{1}{2}+\frac{2}{3}|\zeta|^2}\right)^2+\frac{r^2}{6}\frac{|\dot\zeta|^2}{\left(\frac{1}{2}+\frac{2}{3}|\zeta|^2\right)^2}$$

$\{q_k\}\to\{r,s_k\}=\{$size, other variables$\}$

$\Rightarrow$ for N-body, any masses, in any dimension

$$L=\frac{\dot r^2}{2}+\frac{r^2}{2}f(\dot s_k,s_k)+U(r,s_k)$$

Saari's relation

$$L=\frac{\dot r^2}{2}+\frac{r^2}{2}f(\dot s_k,s_k)+U(r,s_k)\quad,\qquad E=\frac{\dot r^2}{2}+\frac{r^2}{2}f-U$$

then

$$\frac{d}{dt}\frac{\partial L}{\partial\dot r}=\frac{\partial L}{\partial r}\quad\text{and}\quad\dot E=0$$

$$\Rightarrow\ \frac{d}{dt}\left(\frac{r^4}{2}f\right)=r^2\left(\dot U-\dot r\frac{\partial U}{\partial r}\right)$$

$$U=\frac{\mu}{\alpha r^\alpha}\Rightarrow rhs=\frac{r^{2-\alpha}}{\alpha}\frac{d\mu}{dt}$$

Therefore, the Saari's relation

$$\frac{d}{dt}\Big(r^2(\text{kinetic energy for shape})\Big)=\frac{r^{2-\alpha}}{\alpha}\frac{d\mu}{dt}$$

is true
for any masses, any dimension
and any Number of bodies

Shape variable

$$\zeta=x+iy\in\mathbb{C}\sim\mathbf{x}=(x,y)\in\mathbb{R}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{x}}}\right)=\frac{\partial L}{\partial\mathbf{x}}$$

$$\Downarrow$$

$$\frac{d^2\mathbf{x}}{ds^2}=\frac{2C-\frac{4}{3}\left(\mathbf{x}\wedge\frac{d\mathbf{x}}{ds}\right)}{\frac{1}{2}+\frac{2}{3}|\mathbf{x}|^2}\left(\frac{dy}{ds},-\frac{dx}{ds}\right)+\frac{3r^{2-\alpha}}{\alpha}\nabla\mu.$$

here $\nabla\mu=\left(\frac{\partial\mu}{\partial x},\frac{\partial\mu}{\partial y}\right)$

Saari's conjecture
for planar
$m_k = 1$,
3-body problem
under $\dfrac{1}{r^\alpha}$ potential

Fujiwara, Fukuda, Ozaki & Taniguchi, 2011

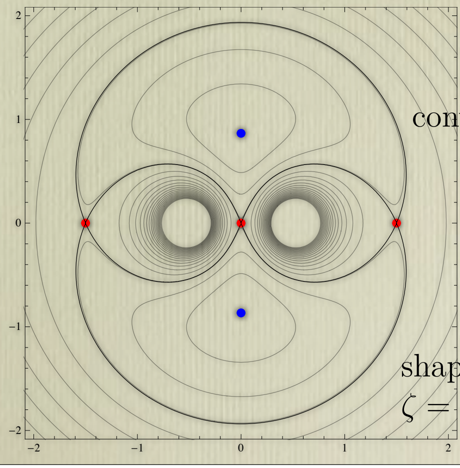
Saari's conjecture for ...

$$m_k = 1,\ U = \sum \frac{1}{r_{ij}^\alpha},\ \mu = I^{\alpha/2} U$$

$\dfrac{d\mu}{dt} = 0$

then

$\dfrac{d\mathbf{x}}{dt} = 0$



contour of $\mu(\mathbf{x})$

shape variable:
$\zeta = x + iy \sim (x, y)$

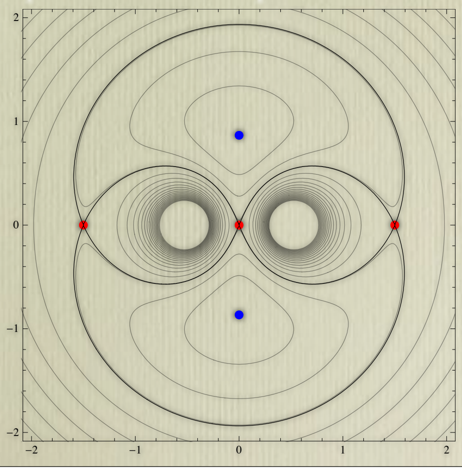
We will show that ...

motion along a contour is
not compatible to the equations of motion

$\dfrac{d\mu}{dt} = 0$

then

$\dfrac{d\mathbf{x}}{dt} = 0$



Saari's relation

eq. of motion for $r \Rightarrow \dfrac{d}{ds}\left(\dfrac{1}{6}\left|\dfrac{d\mathbf{x}}{ds}\right|^2\right) = \dfrac{r^{2-\alpha}}{\alpha}\dfrac{d\mu}{ds}$

therefore $\dfrac{d\mu}{ds} = 0$ yields ...

$\downarrow$

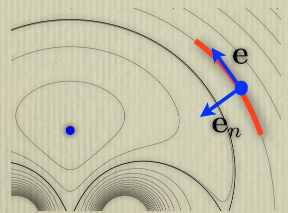
$\left|\dfrac{d\mathbf{x}}{ds}\right| = k =$ constant ≥ 0

$\downarrow$

Saari's conjecture claims that
only $k = 0$ is possible.

if motion with $\dfrac{d\mu(\mathbf{x})}{ds} = 0$ and $\left|\dfrac{d\mathbf{x}}{ds}\right| = k > 0$ exists ...

$|\mathbf{e}| = |\mathbf{e}_n| = 1,$
$\mathbf{e} \cdot \mathbf{e}_n = 0, \mathbf{e} \wedge \mathbf{e}_n = 1.$



motion along a contour
with constant speed

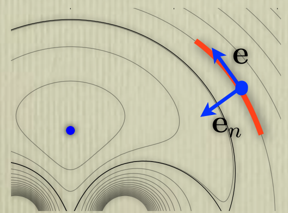
$$\frac{d\mathbf{x}}{ds} = \epsilon k \mathbf{e} \qquad \epsilon = \pm 1$$

$$\frac{d^2\mathbf{x}}{ds^2} = \frac{k^2}{\rho}\mathbf{e}_n$$

ρ: radius of the curvature

if motion with $\dfrac{d\mu(\mathbf{x})}{ds} = 0$ and $\left|\dfrac{d\mathbf{x}}{ds}\right| = k > 0$ exists ...

$|\mathbf{e}| = |\mathbf{e}_n| = 1,$
$\mathbf{e} \cdot \mathbf{e}_n = 0, \mathbf{e} \wedge \mathbf{e}_n = 1.$



motion along a contour
with constant speed

$$\frac{d\mathbf{x}}{ds} = \epsilon k \mathbf{e} = \frac{\epsilon k}{|\nabla\mu|}(-\mu_y, \mu_x), \ \epsilon = \pm 1$$

$$\frac{d^2\mathbf{x}}{ds^2} = \frac{k^2}{\rho}\mathbf{e}_n = \frac{k^2}{\rho}\frac{1}{|\nabla\mu|}\nabla\mu,$$

$$\frac{1}{\rho} = \frac{\epsilon}{|\nabla\mu|}(\mu_y^2\mu_{xx} - 2\mu_x\mu_y\mu_{xy} + \mu_x^2\mu_{yy})$$

Motion is determined by μ completely.

we have two expressions for $\dfrac{d^2\mathbf{x}}{ds^2}$

from $\mu = $ constant,

and from equation of motion

They must be the same

$$\Rightarrow \frac{2C - \frac{4}{3}\frac{\epsilon k}{|\nabla\mu|}(\mathbf{x}\cdot\nabla\mu)}{\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2}\frac{\epsilon k}{|\nabla\mu|} + \frac{3r^{2-\alpha}}{\alpha} = \frac{k^2}{\rho}\frac{1}{|\nabla\mu|}$$

C: angular momentum,
k: constant speed
$\epsilon = \pm 1$: direction of the motion

Let α =2. Because...

$$\frac{2C - \frac{4}{3}\frac{\epsilon k}{|\nabla\mu|}(\mathbf{x}\cdot\nabla\mu)}{\frac{1}{2} + \frac{2}{3}|\mathbf{x}|^2}\frac{\epsilon k}{|\nabla\mu|} + \frac{3r^{2-\alpha}}{\alpha} = \frac{k^2}{\rho}\frac{1}{|\nabla\mu|}$$

For $\alpha = 2$, this term is constant.

This equation gives a condition for x and y,

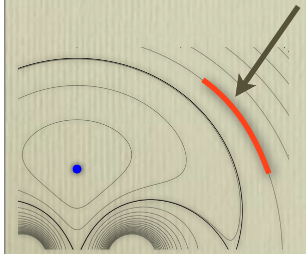
$$P(x^2, y^2, C^2, k^2) = 0$$

polynomial x^{60}, y^{60}, C^2 and k^4,

independent from r

on the other hand, $\mu = \mu_0$ gives another condition for $\alpha = 2$,

$$
\begin{aligned}
Q(x^2, y^2, \mu_0) =& 27 + 64x^6 + 156y^2 + 208y^4 + 64y^6 \\
&+ 48x^4\left(3 + 4y^2\right) + 4x^2\left(27 + 88y^2 + 48y^4\right) \\
&- 6\mu_0\left(16x^4 + 8x^2\left(-1 + 4y^2\right) + \left(1 + 4y^2\right)^2\right) \\
=& 0.
\end{aligned}
$$



we can prove finite arc $Q = 0$ must have finite interval of x

$$\Rightarrow \quad P(x^2, y^2, C^2, k^2) = 0 \quad \text{and} \quad Q(x^2, y^2, \mu_0) = 0$$

must be satisfied in an finite interval of x

$\Downarrow$

eliminating y using

$\text{Resultant}[P, Q, y^2] = R(x^2, C^2, k^2, \mu_0)$

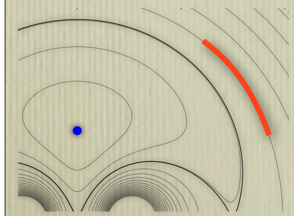
$\Downarrow$

$$R = Ax^8(4x^2 - 1)^6 \sum_{0 \le n \le 24} c_n(C^2, k^2, \mu_0) x^{2n}$$

$= 0$ for all x

$\Downarrow$

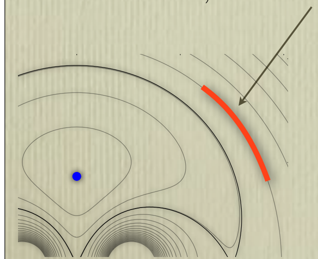
25 conditions $c_n = 0$
for only 3 parameters C^2, k^2 and μ_0.



25 conditions $c_n = 0$
for only 3 parameters C^2, k^2 and μ_0.

actually, we can show that
there are no parameters that satisfy
$c_{24} = c_{23} = c_{22} = 0$

therefore, no such arc with $P = Q = 0$



namely, motion $\dfrac{d\mu}{ds} = 0$

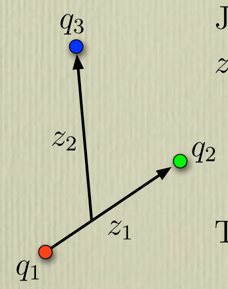
with $\left|\dfrac{d\mathbf{x}}{ds}\right| > 0$ is impossible

$\therefore \left|\dfrac{d\mathbf{x}}{ds}\right| = 0.$ This completes a proof of the Saari's conjecture

Summary

The Shape variable

Moeckel and Montgomery, 2007



Jacobi variables:

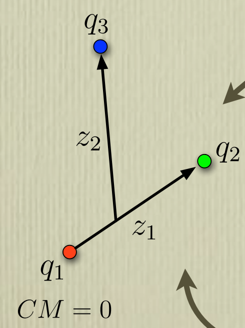
$z_1 = q_2 - q_1,\ z_2 = q_3 - \dfrac{m_1 q_1 + m_2 q_2}{m_1 + m_1}.$

The Shape variable: $\zeta = \dfrac{z_2}{z_1}.$

$q_k, z_i, \zeta \in \mathbb{C}$

Shape variable

The Shape variable: $\zeta = \dfrac{z_2}{z_1}.$

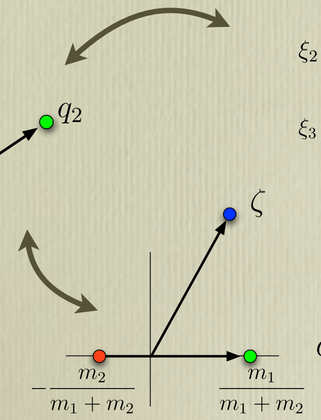


$$\xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{m_2}{m_1 + m_2} - \frac{m_3}{M}\zeta,$$

$$\xi_2 = \frac{q_2}{q_2 - q_1} = \frac{m_1}{m_1 + m_2} - \frac{m_3}{M}\zeta,$$

$$\xi_3 = \frac{q_2}{q_2 - q_1} = \frac{m_1 + m_2}{M}\zeta,$$

$CM = 0$



$CM = \dfrac{m_3}{M}\zeta$

Lagrangian

$$q_k = re^{i\theta}\frac{\xi_k}{\sqrt{\sum m_k|\xi_k|^2}} \Rightarrow L = \frac{K}{2} + \frac{U}{\alpha}$$

$$\frac{K}{2} = \sum m_k|\dot{q}_k|^2 = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\left(\dot{\theta} + \frac{(m_1+m_2)m_3}{M}\frac{\zeta \wedge \dot{\zeta}}{n}\right)^2 + \frac{m_1 m_2 m_3}{M}\frac{r^2|\dot{\zeta}|^2}{2n^2}$$

$$U = \frac{\mu(\zeta)}{r^\alpha}$$

Moeckel and Montgomery, 2007

$$n = \frac{m_1 m_2}{m_1 + m_2} + \frac{(m_1+m_2)m_3}{M}|\zeta|^2$$

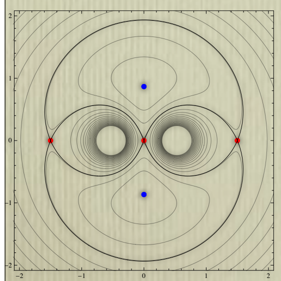
$$\mu = n^{\alpha/2}\left(m_1 m_2 + \frac{m_2 m_3}{|\frac{m_1}{m_1+m_2} - \zeta|^\alpha} + \frac{m_3 m_1}{|\frac{m_2}{m_1+m_2} + \zeta|^\alpha}\right)$$

Saari's conjecture

$\dfrac{d\mu}{dt} = 0 \Rightarrow \dfrac{d\zeta}{dt} = 0$ is proved

for planar 3-body problem with $m_k = 1$ and $\alpha = 2$.

Fujiwara, Fukuda, Ozaki & Taniguchi 2011



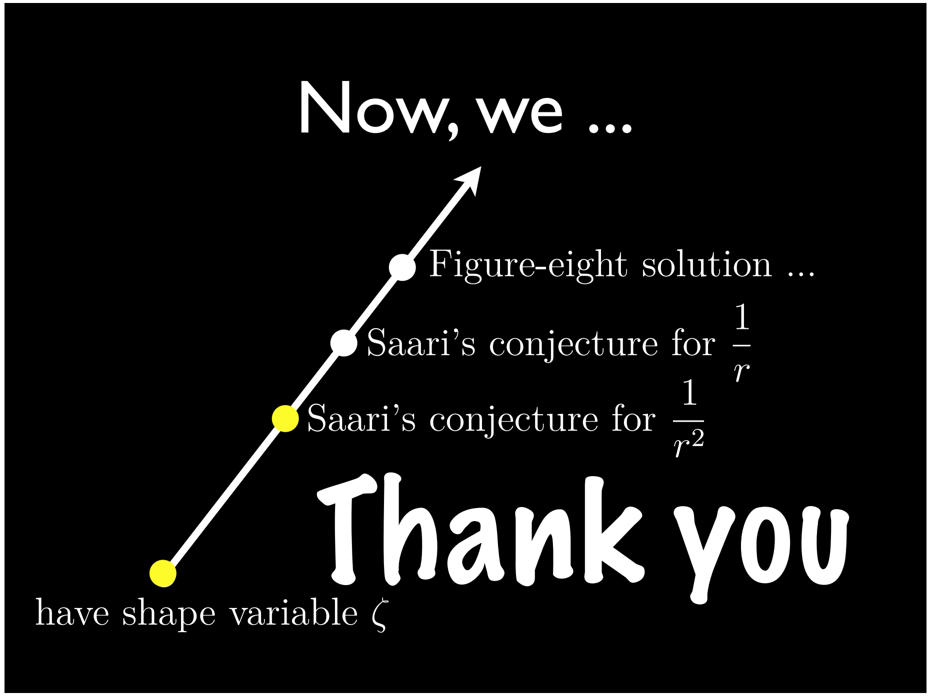
contour of $\mu(\zeta)$

by showing that ...

If $\dfrac{d\mu}{ds} = 0$, the motion is determined by μ,

especially, $\dfrac{d^2\mathbf{x}}{ds^2} = \dfrac{k^2}{\rho}\mathbf{e}_n.$

However, this is
incompatible with the equation of motion.


Now, we ...
Figure-eight solution ...
Saari's conjecture for $\frac{1}{r}$
Saari's conjecture for $\frac{1}{r^2}$
Thank you
have shape variable ζ