Bifurcations of Figure-eight solutions — Bifurcations and Symmetry —

T. Fujiwara, H. Fukuda, H. Ozaki

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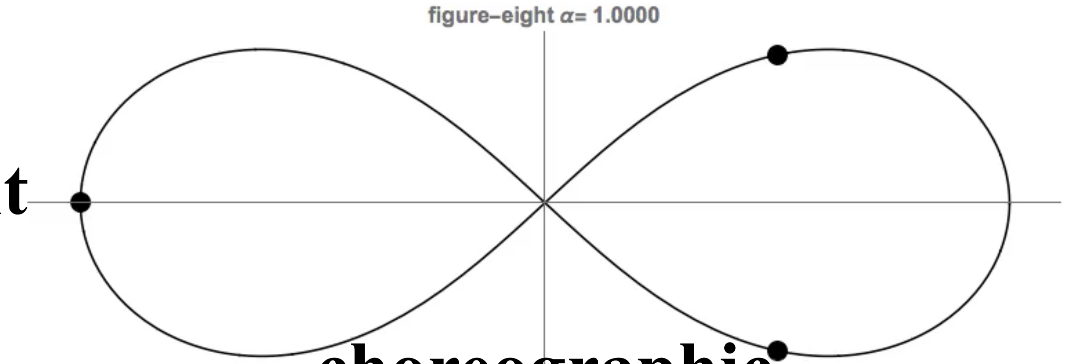
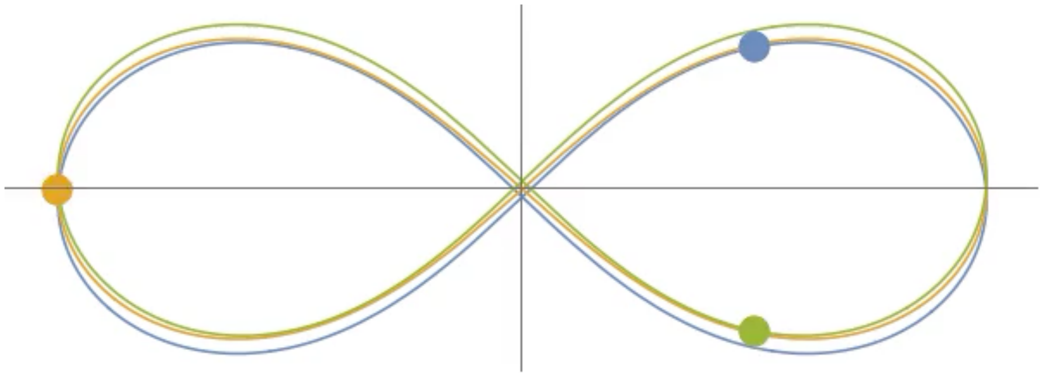
the figure-eight(1993,2000)



the figure-eight

choreographic

$$V = \frac{1}{ar^a},\ a > -2.$$



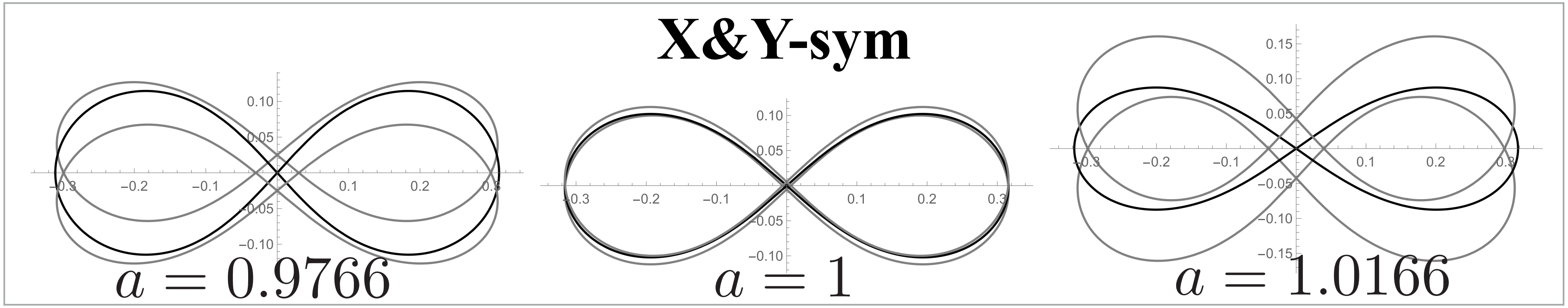
Simó's H

non-choreographic

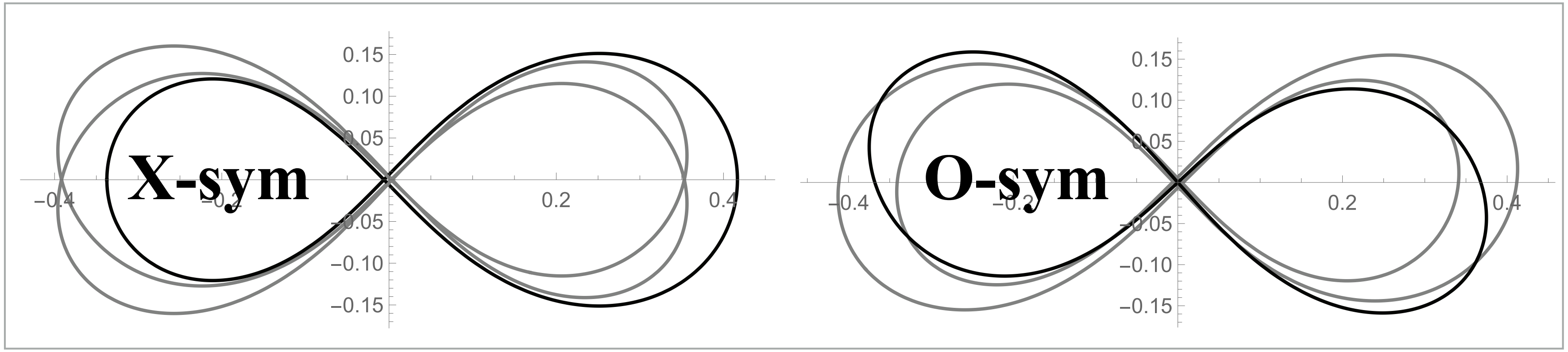
$a = 1$ (Newton potential)

bifurcations

$$V = \frac{1}{ar^a},\ a > -2.$$



Simó's H: $a_0 = 0.9966$ both $a < a_0$ and $a_0 < a$



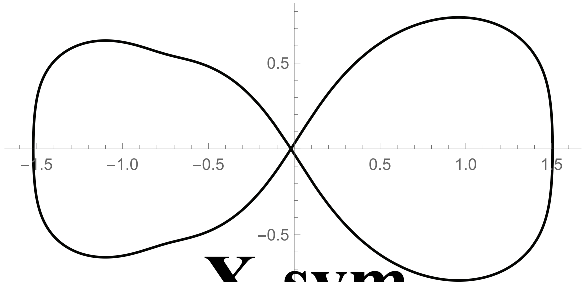
one side: $a_0 = 1.3424 < a$

two different solutions emerge
from this bifurcation point

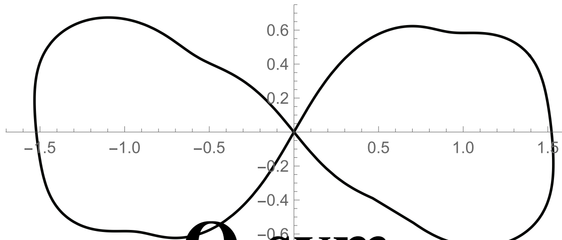
Munõz-Almaraz, Galán-Vioque, Freire, and Vanderbauwhede 2005

Bifurcations of figure-eight solutions Lennard-Jones potential

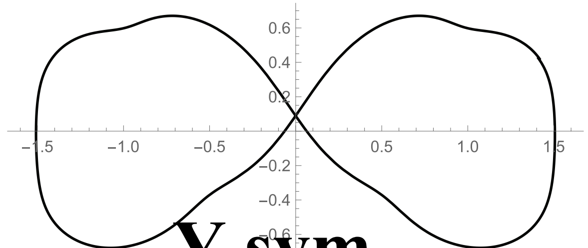
$$V \sim \frac{1}{r^6} - \frac{1}{r^{12}}, T > T_0$$



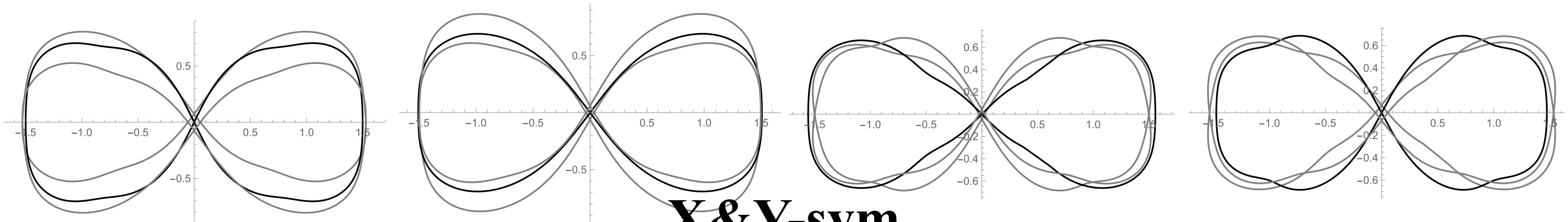
X-sym



O-sym

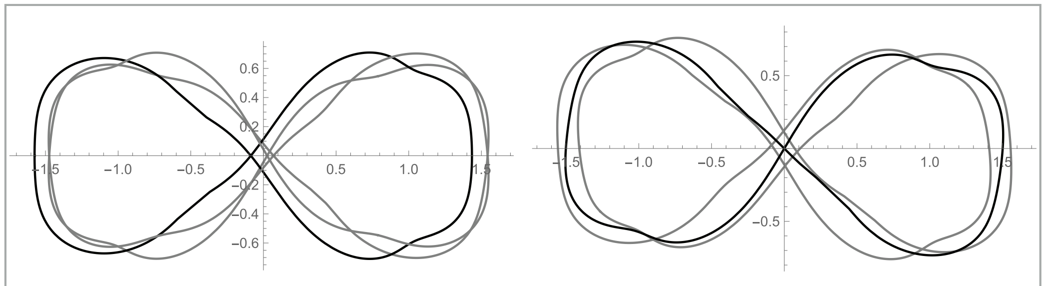


Y-sym



X&Y-sym

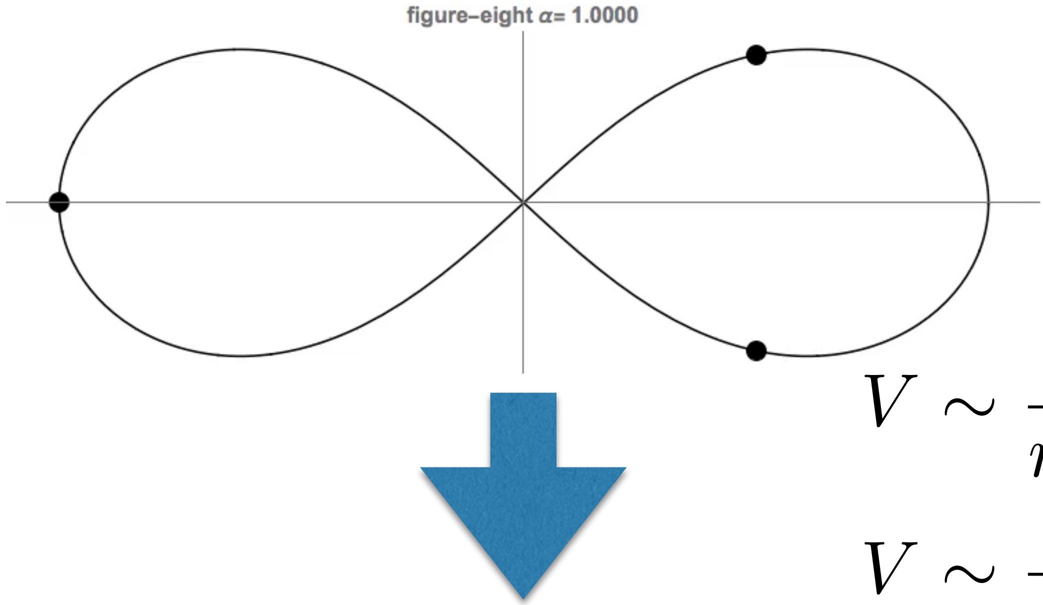
one bifurcation point:



X-sym **O-sym**

Fukuda, Fujiwara and Ozaki 2019

Bifurcations of figure-eight solutions



$$V \sim \frac{1}{r^a},\ a > -2.$$

$$V \sim \frac{1}{r^6} - \frac{1}{r^{12}}, T > T_0$$

6 bifurcation patterns

5 patterns ⇆ Symmetry breaking

Does not depend on potential or parameter

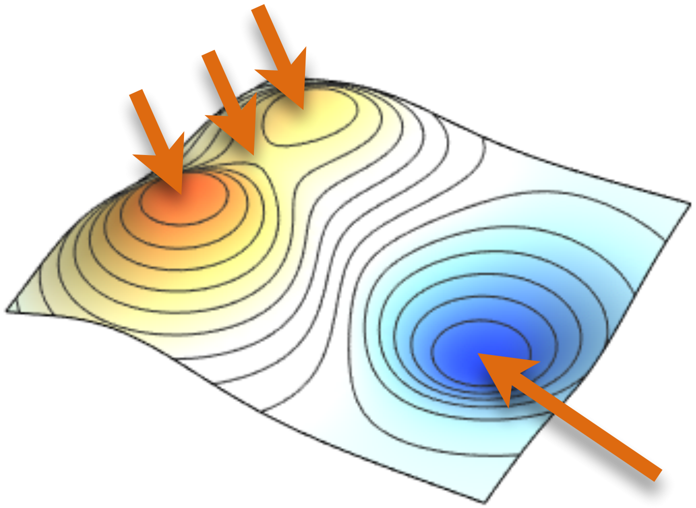
why 6 patterns？ any more or not？

and my big question or a strong desire is …

Who has seen the Action?

Lagrangian and Action: $S = \int dt L$

a stationary point $\delta S = 0$

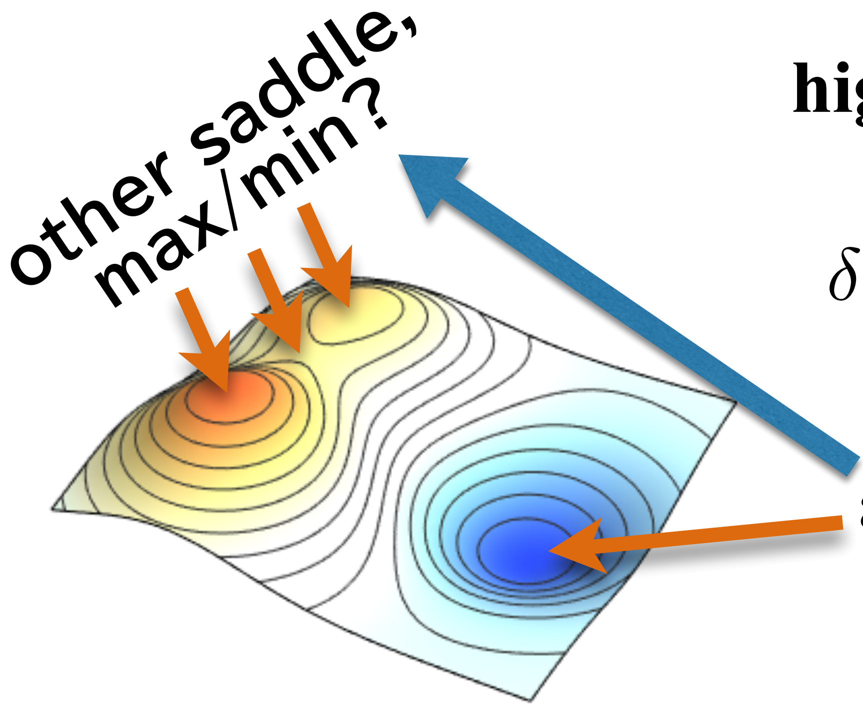


a solution of the equation of motion

variational method: local minimum of S

$\Rightarrow \delta S = 0$

Who has seen the Action?



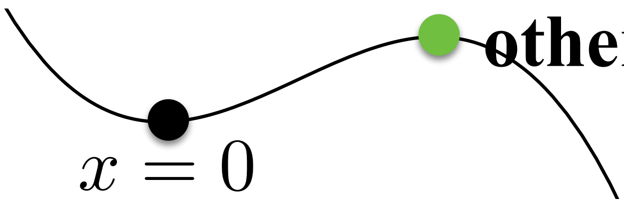
higher derivatives of the action of a solution

$$\delta^2 S(q_\circ), \delta^3 S(q_\circ), \delta^4 S(q_\circ), \ldots$$

action around this solution

if the action were a function of one variable

$$s(x) = s(0) + s'(0)x + s''(0)x^2/2 + s'''(0)x^3/3! + \ldots$$



difficlty: distance to the other solution is large

bifurcatios: distance → 0

Necessary condition

if two stationary points meet

$$\delta S(q_o) = \delta S(q_o + R\Phi) = 0 \to \delta^2 S(q_o) = 0 \text{ for } R \to 0$$

$$\left(\begin{aligned} & m\frac{d^2}{dt^2} q_o = \frac{\partial U(q_o)}{\partial q} \\ & m\frac{d^2}{dt^2}(q_o + R\Phi) = \frac{\partial U(q_o + R\Phi)}{\partial q} \end{aligned} \right.$$

$$= \frac{\partial U(q_o)}{\partial q} + R\frac{\partial^2 U(q_o)}{\partial q^2}\Phi + O(R^2)$$

$$\mathcal{H}\Phi = O(R) \to 0 \text{ for } R \to 0$$

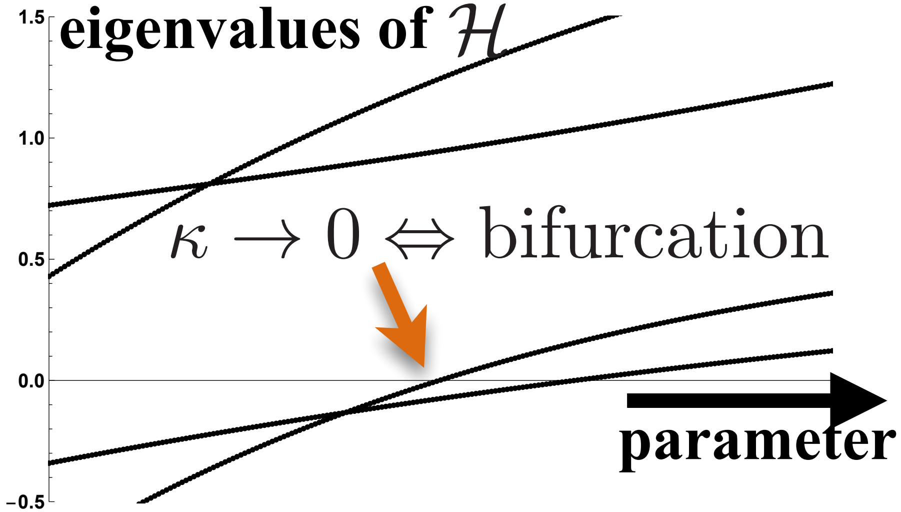
linearised equations of motion

namely, one eigenvalue $\to 0$

$$\mathcal{H} = -m\frac{d^2}{dt^2} + \frac{\partial^2 U}{\partial^2 q}$$

Eigenvalues and eigenfunctions of Hessian

$$\mathcal{H} = \begin{pmatrix} * & & & & & & & \\ & * & & & & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & * & & \\ & & & & * & * & & \\ & & & & & & * & * \\ & & & & & & * & * \end{pmatrix}$$



the eigenvalues and eigenfunctions of $\mathcal{H}$
are classified by
the irreducible representations of G

$$g \in G$$

$$S[gq] = S[q] \text{ for any } q(t+T) = q(t), gq_o = q_o \text{ for solution } q_o$$

Lyapunov-Schmidt reduction

$$\mathcal{H}\phi = \kappa\phi,\ \mathcal{H}\psi_\alpha = \lambda_\alpha\psi_\alpha$$

$$S(r,\epsilon) = S[q_o + r\phi + r\sum_\alpha \epsilon_\alpha\psi_\alpha] - S[q_o]$$

stationary point: $\partial_r S(r,\epsilon) = \partial_\epsilon S(r,\epsilon) = 0$

solution $\epsilon_\alpha = \epsilon_\alpha(r)$

unique because $\lambda_\alpha \neq 0$

$$S_{\mathrm{LS}}(r) = S(r,\epsilon(r))$$

stationary point: $\dfrac{dS_{\mathrm{LS}}(r)}{dr} = 0$

original or bifurcated solution(s)

Group Theoretic Methods in Bifurcation Theory

- **Sattinger 1979**
  - Group theoretic methods in bifurcation theory
- **Golubitsky and Scaeffer 1985**
  - Singularities and groups in bifurcation theory I & II
- **Chenciner, Féjoz, and Montgomery 2004**
  - Rotating Eights: I. The three Γi families.

Irreducible representations of D_n

figure-eight solutions have

$$D_6 = \{1, x, x^2, \ldots, x^5, y, yx, yx^2, \ldots, yx^5, \}$$
$$x^6 = y^2 = 1,\ xy = yx^{-1}.$$

$$\mathcal{H} = \begin{pmatrix} * & & & & & & & \\ & * & & & & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & * & & \\ & & & & * & * & & \\ & & & & & & * & * \\ & & & & & & * & * \end{pmatrix}$$

trivial representation keeps D_6 symmetry

other representations break the symmetry

one dimensional representations

two dimensional representations

Symmetry of the reduced action

$$\text{for } g \in G: \; S[gq] = S[q], gq_o = q_o$$

for one dimensional representations

$$S_{LS}(r) : \text{a function of } r$$

$$\text{if } g\phi = -\phi \Rightarrow S_{LS}(-r) = S_{LS}(r)$$

for two dimensional representations

$$S_{LS}(r, \theta) : \text{a function of } r \text{ and } \theta$$

$$r\phi(\theta) = r\cos(\theta)\phi_1 + r\sin(\theta)\phi_2$$

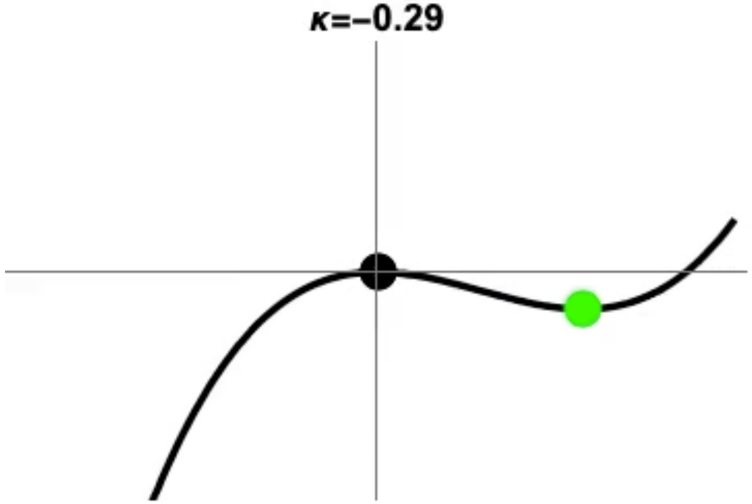
$$\text{if } g(r\phi(\theta)) = r'\phi(\theta') \rightarrow S_{LS}(r', \theta') = S_{LS}(r, \theta)$$

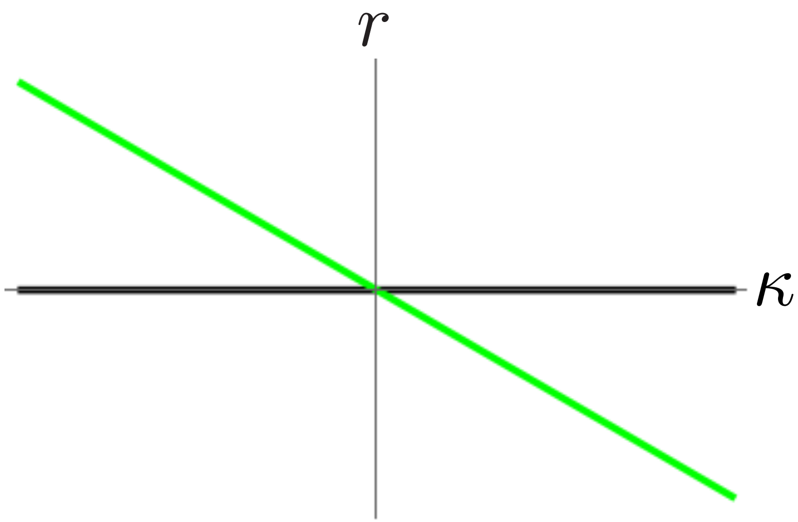
bifurcations

for trivial representation

κ is not degenerate

$$S_{LS}(r) = \frac{\kappa}{2}r^2 + \frac{A_3}{3!}r^3 + O(r^4),\ A_3 \neq 0$$





example: $A_3 > 0$

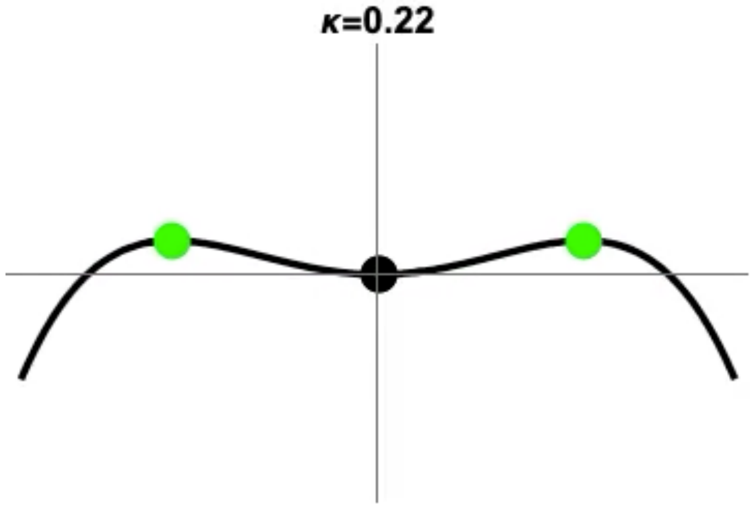
bifurcations

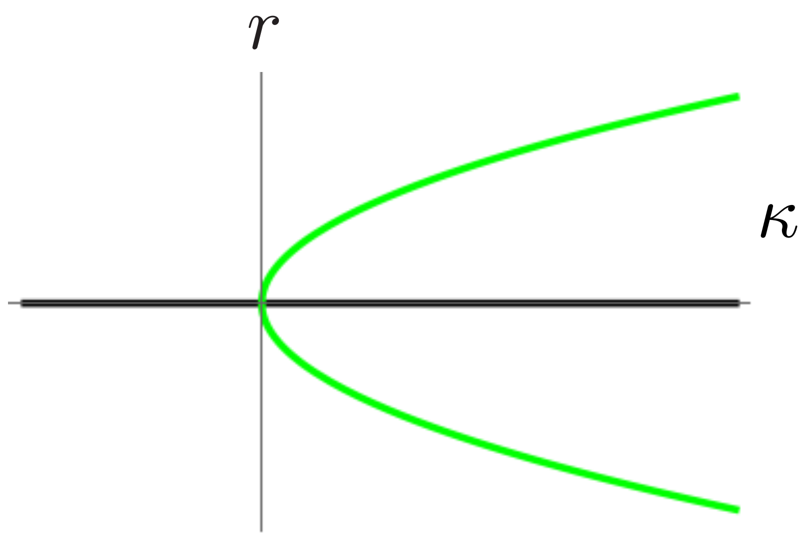
for other one dimensional representations

κ is not degenerate

$$S_{LS}(-r) = S_{LS}(r)$$

$$S_{LS}(r) = \frac{\kappa}{2} r^2 + \frac{A_4}{4!} r^4 + O(r^6),\ A_4 \neq 0,\ A_3 = A_5 = \cdots = 0.$$





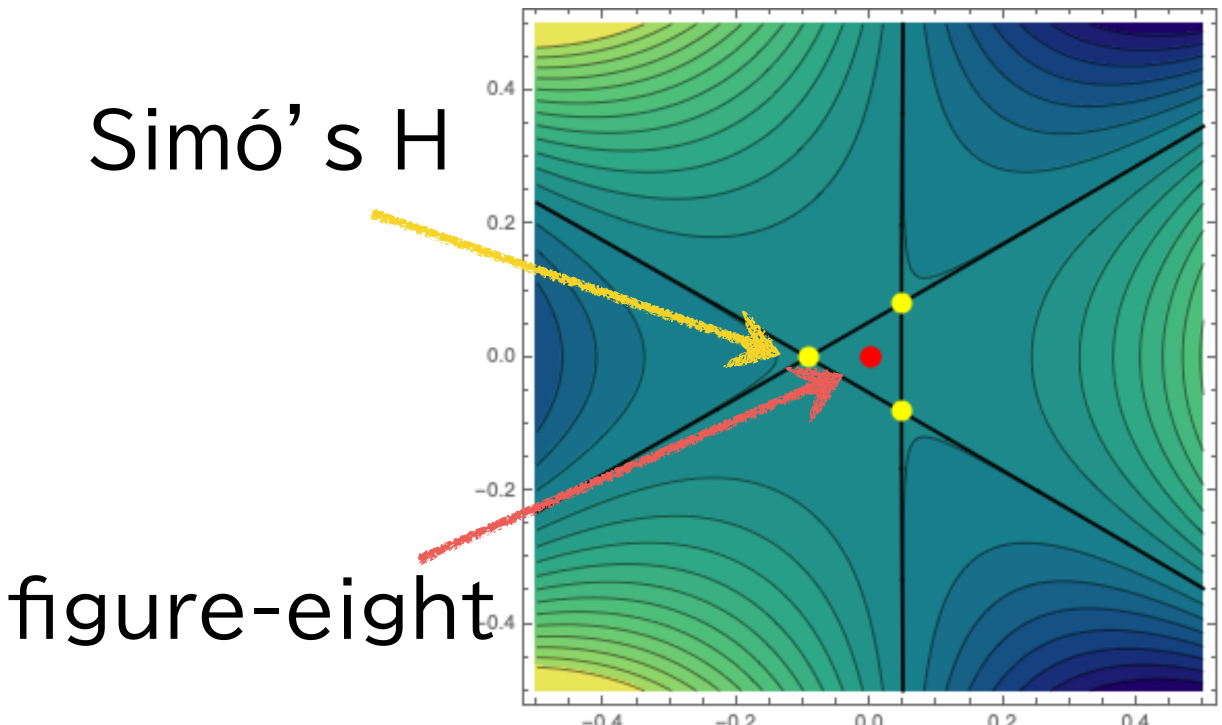
example: $A_4 < 0$

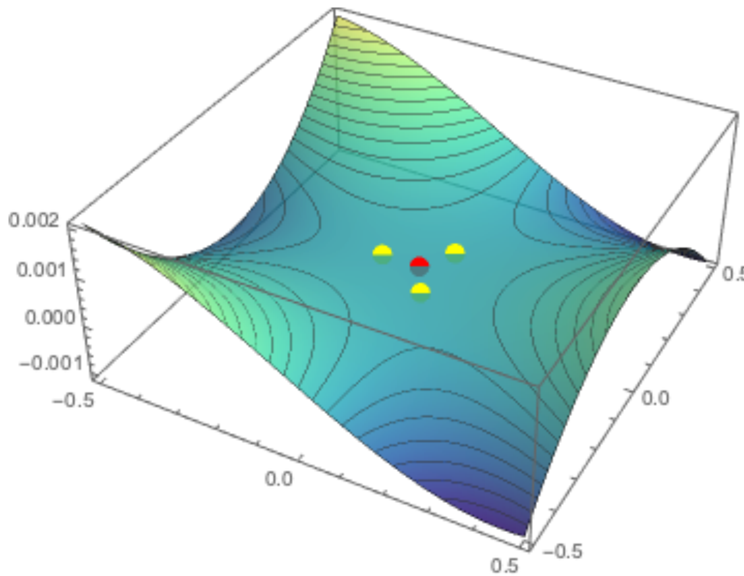
a two dimensional representation

κ is doubly degenerate

$$D_3 : S_{LS}(r, \theta \pm 2\pi/3) = S_{LS}(r, \theta), S_{LS}(r, -\theta) = S_{LS}(r, \theta).$$

$$S_{\mathrm{LS}}(r, \theta) = \frac{\kappa}{2} r^2 + \frac{A_3}{3!} r^3 \cos(3\theta) + \frac{A_4}{4!} r^4 + O(r^5).$$

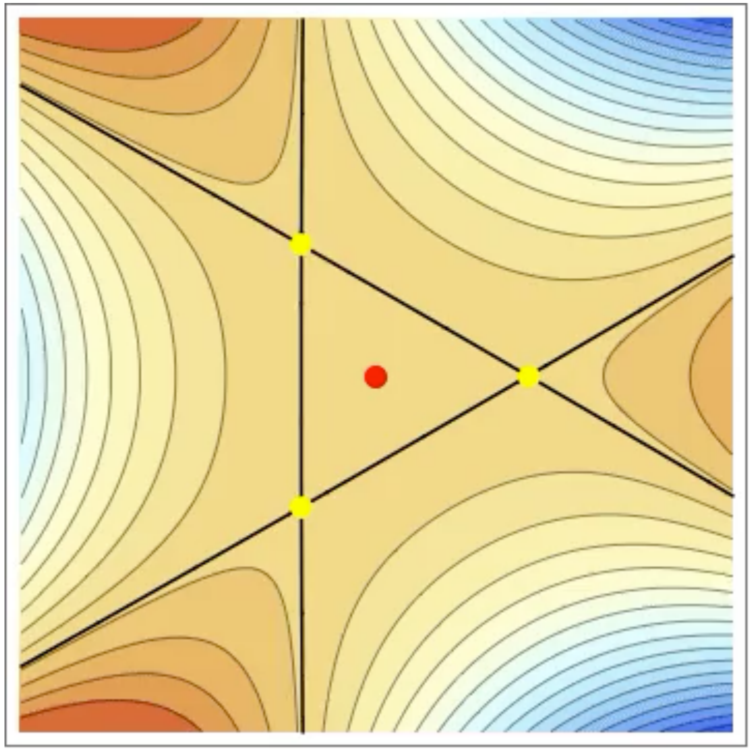




bifurcation of the figure-eight

$$D_3 : S_{LS}(r, \theta \pm 2\pi/3) = S_{LS}(r, \theta), S_{LS}(r, -\theta) = S_{LS}(r, \theta).$$

$$S_{\mathrm{LS}}(r, \theta) = \frac{\kappa}{2} r^2 + \frac{A_3}{3!} r^3 \cos(3\theta) + \frac{A_4}{4!} r^4 + O(r^5).$$



$a_0 = 0.9966$

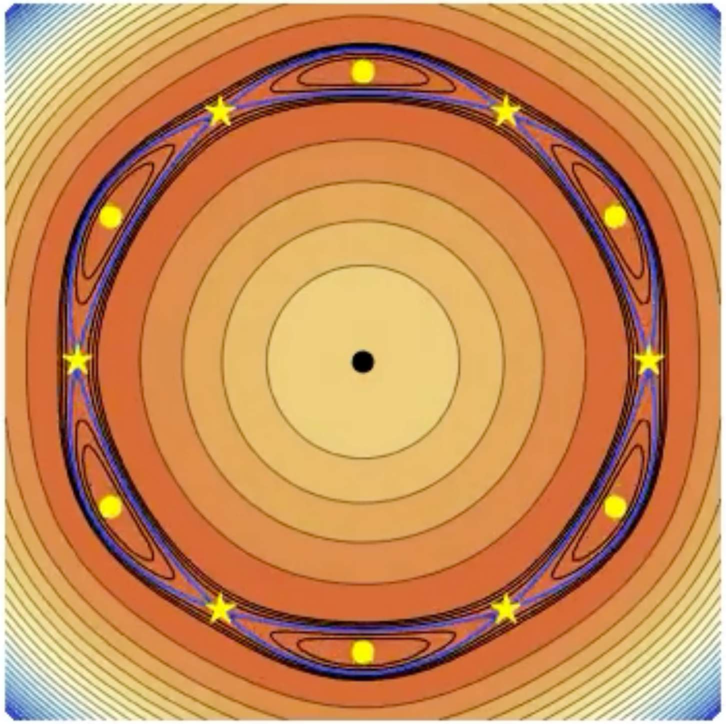
● :figure-eight, ● :Simó's H: saddle

another two dimensional representation

$$D_6 : S_{LS}(r, \theta + 2\pi k/6) = S_{LS}(r, \theta),\ S_{LS}(r, -\theta) = S_{LS}(r, \theta)$$

the faithful representation of D_6

$$S_{\rm LS}(r,\theta) = \frac{\kappa}{2} r^2 + \frac{A_4}{4!} r^4 + \frac{1}{3!} r^3 \Big(A_{6+} \cos(3\theta)^2 + A_{6-} \sin(3\theta)^2 \Big) + O(r^8)$$

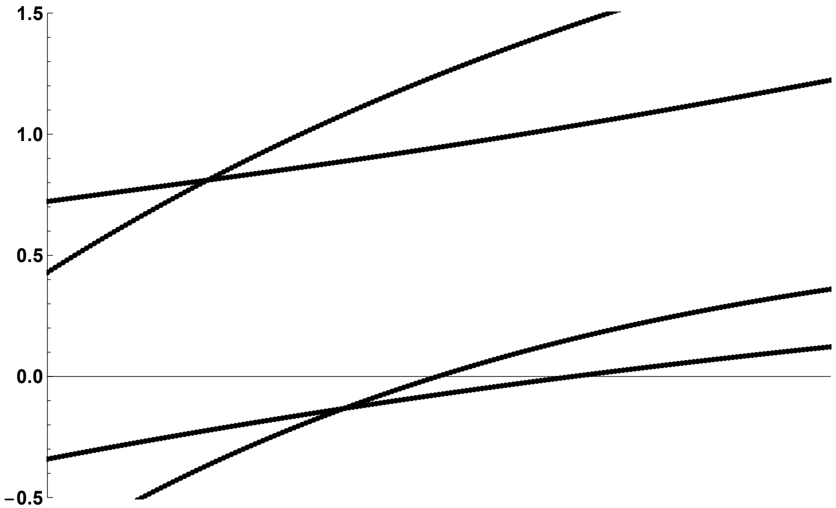


$a_0 = 1.34$

●:figure-eight, ●:local max, ★:saddle

D_6: six irreducible representations and bifurcations of figure-eights

$$\mathcal{H} = \begin{pmatrix} * & & & & & & & \\ & * & & & & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & * & & \\ & & & & * & * & & \\ & & & & & & * & * \\ & & & & & & * & * \end{pmatrix}$$



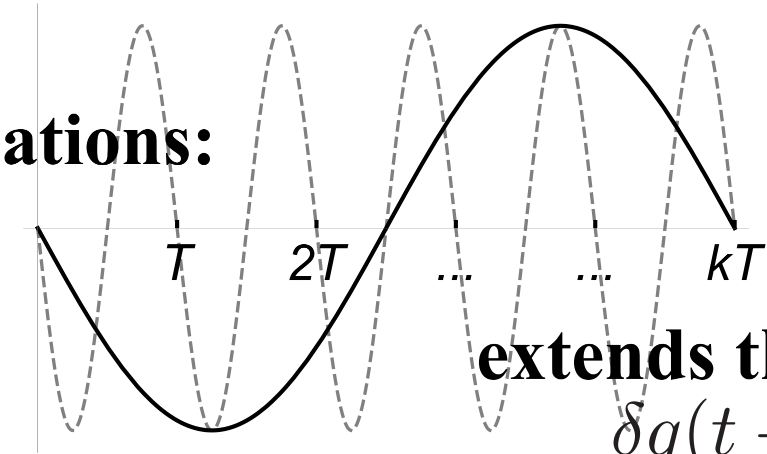
Pattern	${\mathcal{P}_\mathcal{C}}'$	$\mathcal{M}'$	$\mathcal{S}'$	d	a for U_h	T for U_{LJ} α_-	α_+	Symmetry	Type
I	1	1	1	1		14.479	14.479	X- and Y-axis	fold
II	1	1	−1	1			17.132	Y-axis	one-side
III	1	−1	1	1			18.615	O[b]	one-side
IV	1	−1	−1	1	−0.2142	14.595		X-axis	one-side
V	0	1	±1	2	0.9966	14.836	16.878	X- and Y-axis	both-sides
VI	0	−1	±1	2	1.3424	14.861	16.111	O[b] or[a] X-axis	double[a] one-side

Period k bifurcations of figure-eight solutions

for orignal: $D_6 = \langle x, y : x^6 = y^2 = 1,\, xy = yx^{-1} \rangle$

$$x^6 q(t) = q(t+T) \Rightarrow x^6 = 1$$

for period k bifurcations:



extends the functions space

$$\delta q(t+kT) = \delta q(t)$$

$$x^{6k} q(t) = q(t+kT) \Rightarrow x^{6k} = 1$$

$$D_{6k} = \langle x, y : x^{6k} = y^2 = 1,\, xy = yx^{-1} \rangle$$

for period 5 bifurcations of figure-eights

$$D_{30} = \langle x, y : x^{30} = y^2 = 1,\, xy = yx^{-1} \rangle$$

period k bifurcations of D_1

$$D_1 = \{1, S\}, Sq(t) \sim q(-t)$$

$$Rq(t) = q(t+T)$$

for period k bifurcations: $\delta q(t+kT) = \delta q(t) \Rightarrow R^k = 1$

$$D_k = \{R, S : R^k = S^2 = 1,\ RS = SR^{-1}\}$$

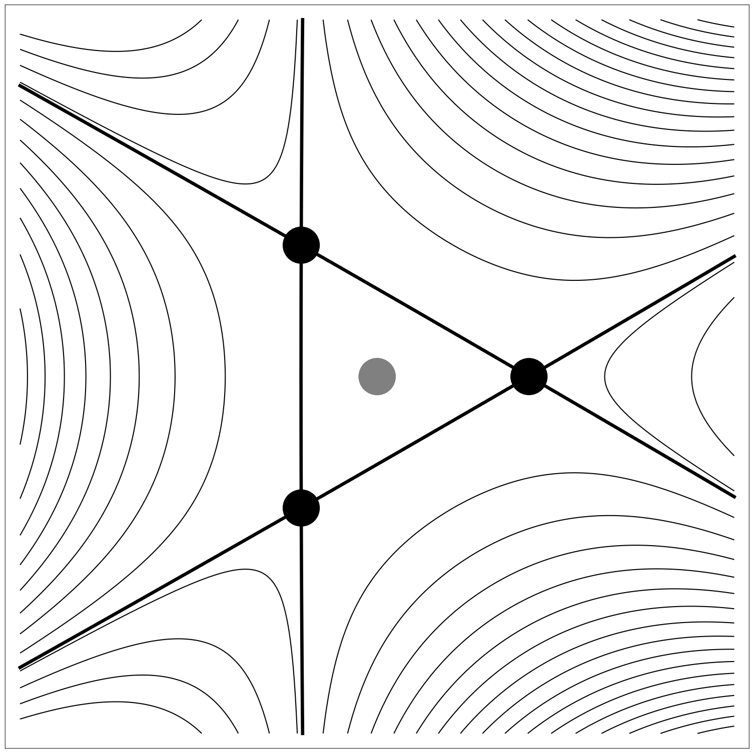
***R* symmetry is broken**

period *k* bifurcation

period 3 bifurcations of D_1

$$D_3 = \{R, S : R^3 = S^2 = 1,\ RS = SR^{-1}\}$$

$$S_{\mathrm{LS}}(r,\theta) = \frac{\kappa}{2}r^2 + \frac{A_3}{3!}r^3\cos(3\theta) + \frac{A_4}{4!}r^4 + O(r^5).$$

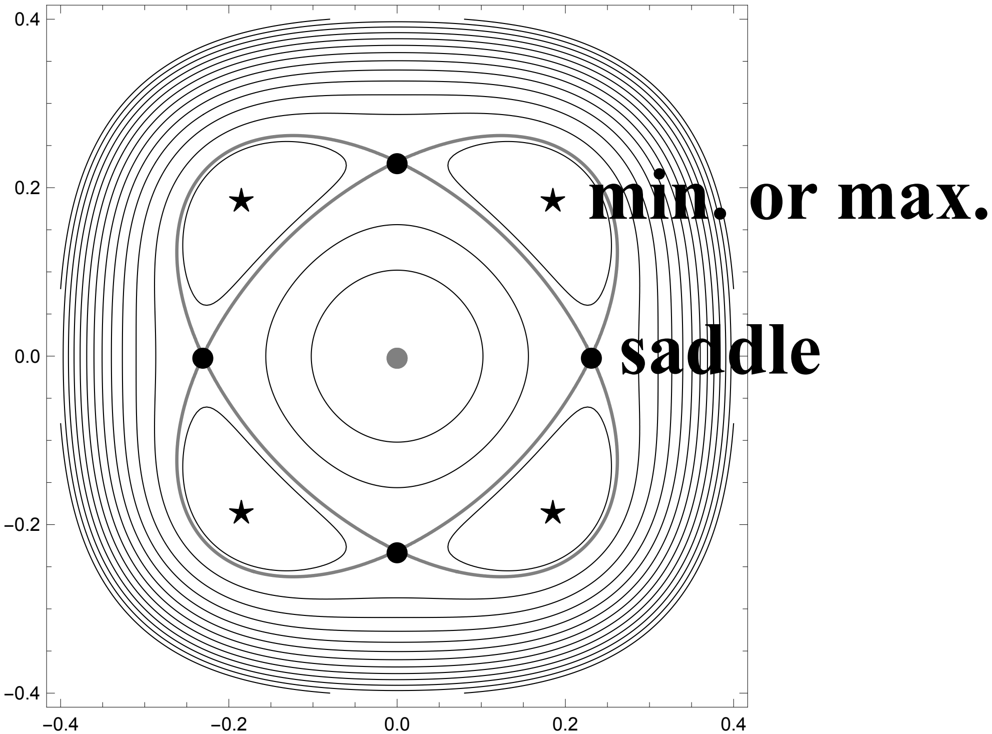


period 4 bifurcations of D_1

$$D_4 = \{R, S : R^4 = S^2 = 1,\ RS = SR^{-1}\}$$

$$S_{\text{LS}}(r,\theta) = \frac{\kappa}{2} r^2 + \frac{r^4}{4!}\underline{(a_4 + b_4 \cos(4\theta))} + O(r^6).$$

case 1:



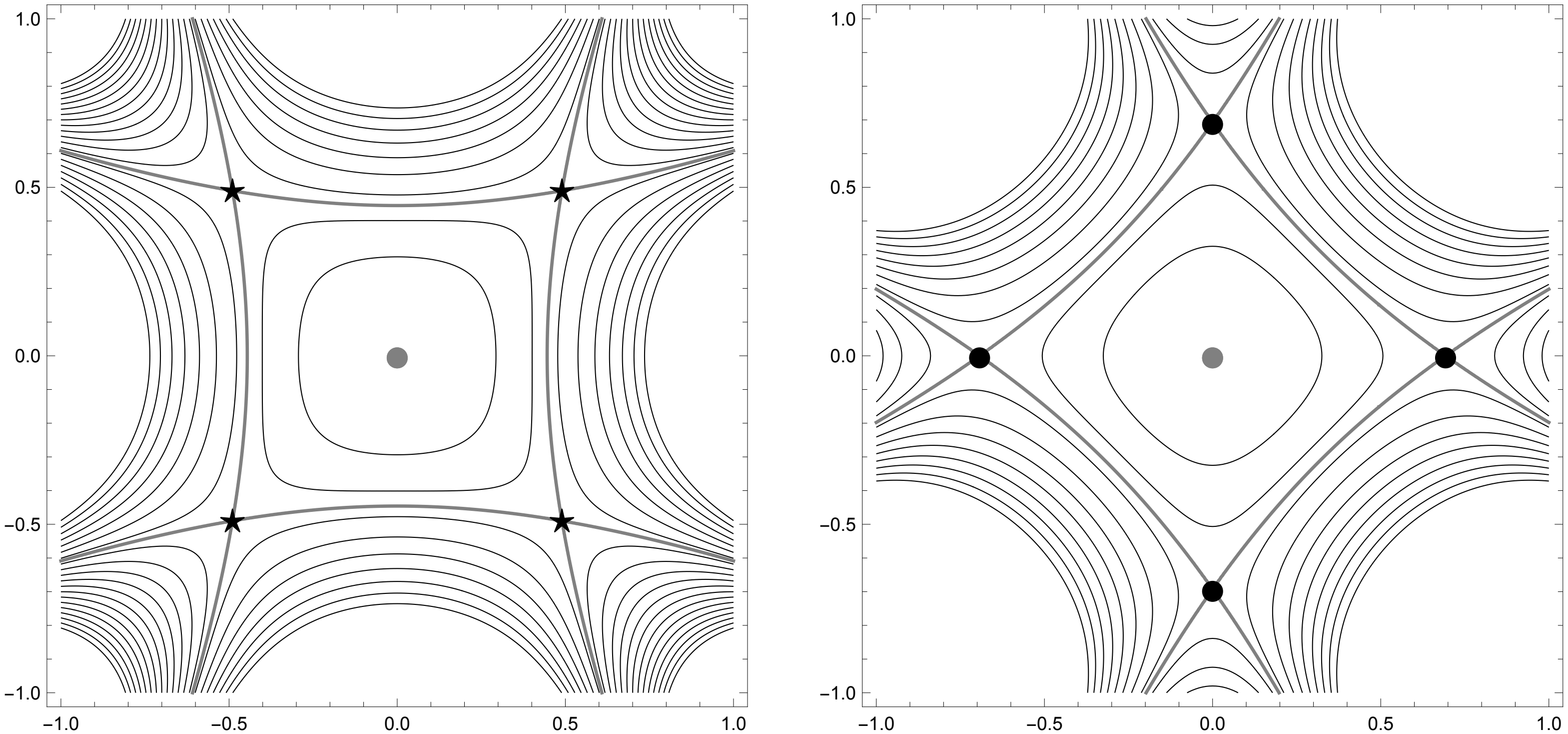
one side

period 4 bifurcations of D_1

$$D_4 = \{R, S : R^4 = S^2 = 1,\ RS = SR^{-1}\}$$

$$S_{\mathrm{LS}}(r,\theta) = \frac{\kappa}{2}r^2 + \frac{r^4}{4!}(a_4 + b_4\cos(4\theta)) + O(r^6).$$

case 2:



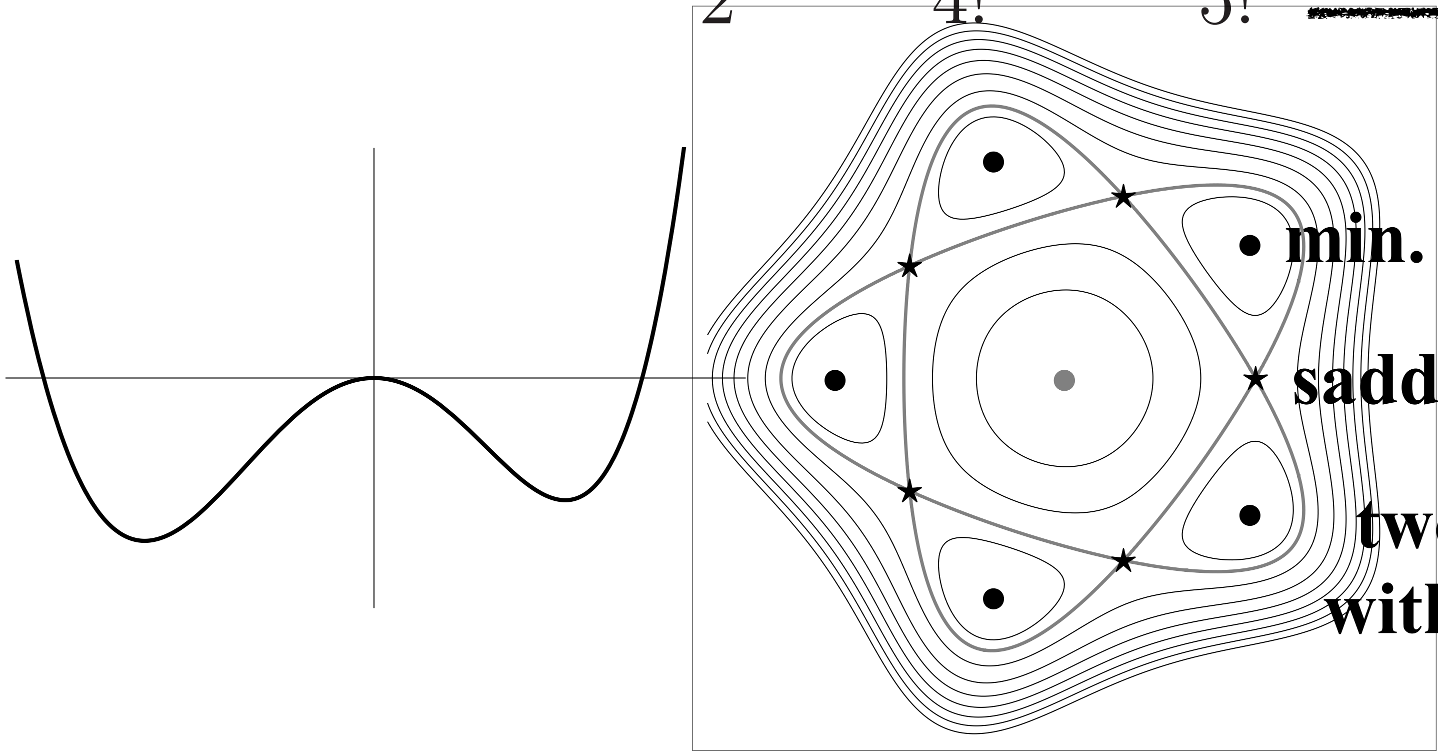
one side **the other side**

bifurcated solutions are saddle

period 5 bifurcations of D_1

$$D_5 = \{R, S : R^5 = S^2 = 1,\ RS = SR^{-1}\}$$

$$S_{\mathrm{LS}}(r,\theta) = \frac{\kappa}{2}r^2 + \frac{A_4}{4!}r^4 + \frac{A_5}{5!}\cos(5\theta) + O(r^6).$$



Summary

Variational principle + group theory

$$S_{LS}(r,\theta) = S[q_o + r\phi(\theta) + r\sum \epsilon_\alpha(r,\theta)\psi_\alpha] - S[q_o]$$

$$\mathcal{H}\phi(\theta) = \kappa\phi(\theta) : \kappa \to 0 \Leftrightarrow \text{a bifurcation}$$

Irreducible representations of group G **determine bifurcation patterns**

Hessian & Lyapunov-Schmidt reduced action

Symmetry: $g(r\phi(\theta)) = r'\phi(\theta') \Rightarrow S_{LS}(r',\theta') = S_{LS}(r,s)$

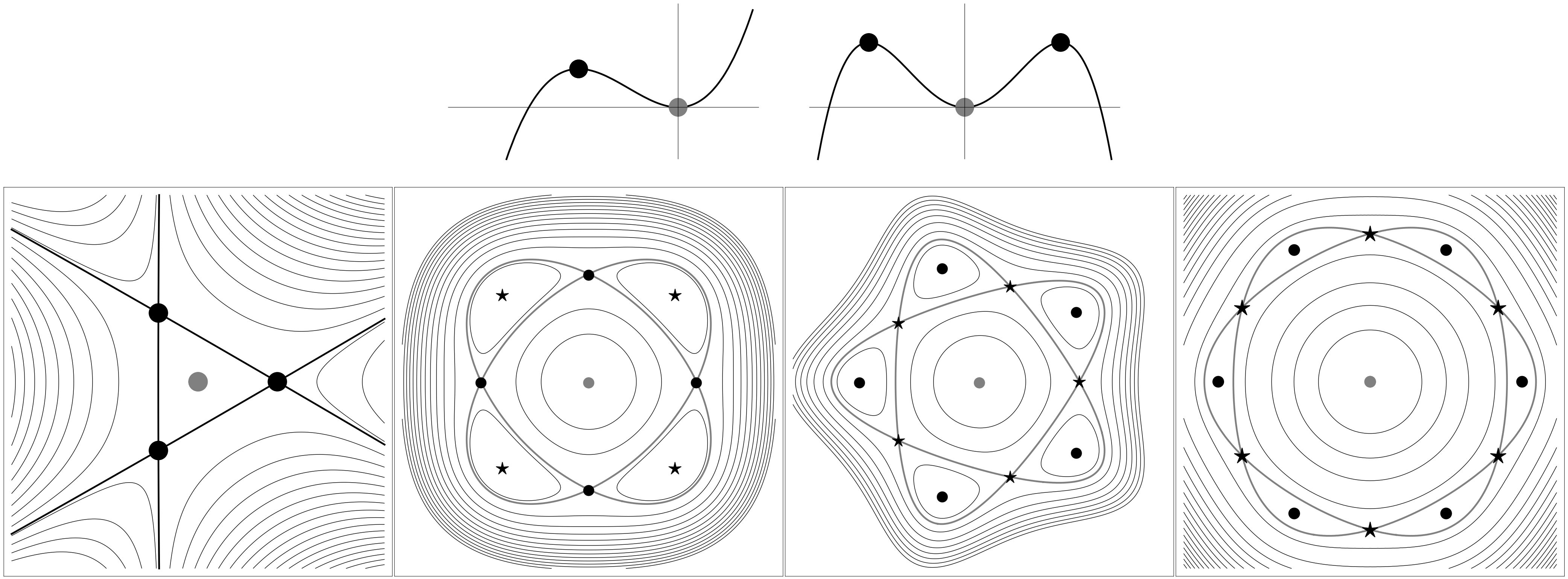
Bifurcations & Symmetry breaking

Explains bifurcations of figure-eights, period k **bifurcations of** D_1

no considerations for stability

Who has seen the Action?

Neither I nor you:
But near the bifurcation points
You and I are seeing the Lyapunov-Schmidt reduced Action



be careful! it is NOT full action

Who has seen the Wind?

by CHRISTINA ROSSETTI

Who has seen the wind?
Neither I nor you:
But when the leaves hang trembling,
The wind is passing through.

Who has seen the wind?
Neither you nor I:
But when the trees bow down their heads,
The wind is passing by.

the action of the Simó's H is greater than that of the figure-eight

the values of action for $T = 1$

Figure 8
13.2077823369 941007626 (*1)
13.2077823369 941007252 (*2)
13.2077823369 940973414 (*3)
Simo H
13.2077837668 871694251 (*1)
13.2077837668 871694248 (*2)
13.2077837668 871692387 (*3)

$$\frac{S_{\text{Simó's H}} - S_{\text{figure-eight}}}{S_{\text{figure-eight}}} = 1.1 \times 10^{-7}$$

*1 and *2 are directly calculated by the integration along the numerical solution

$$S = \int_0^T dt \left(\sum_{k=1,2,3} \frac{1}{2} |\dot{q}_k|^2 + \sum_{i<j} \frac{1}{r_{ij}} \right).$$

*3 is calculated by $S = -3ET$, where E is the energy and $T = 1$ is the period. Since this is determined by the initial conditions, accuracy of this value is high.