Linear stability and Morse index for the figure-eight and k=5 slalom solutions under homogeneous potential

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Three-body choreographies and Continuations

$$L = \frac{1}{2}\sum_k \left|\frac{dq_k}{dt}\right|^2 + \frac{1}{\alpha}\sum_{i,j}\frac{1}{|q_i - q_j|^\alpha}$$

$\alpha = 1$: Newton potential

$$q_0(t) = q(t), q_1(t) = q(t + T/3), q_2(t) = q(t + 2T/3)$$

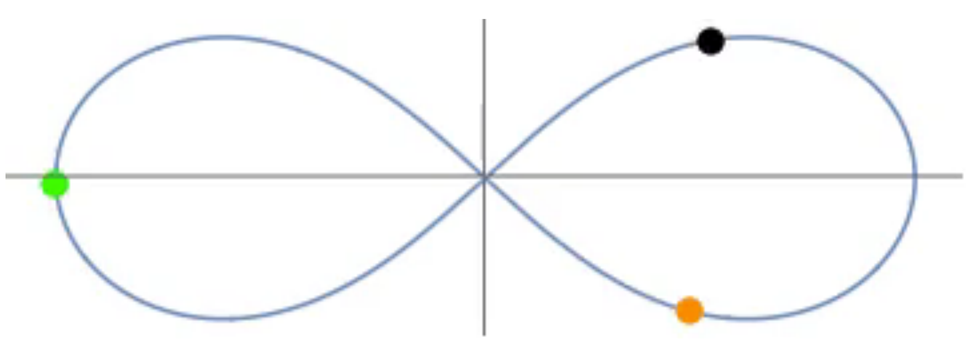


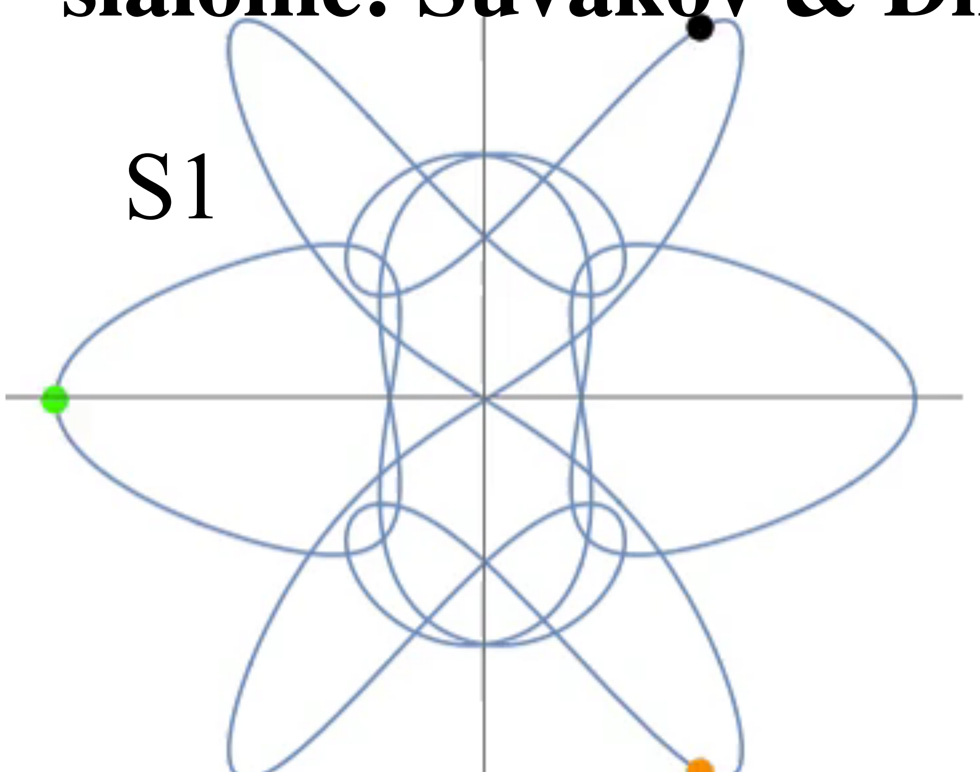
figure-eight solution
C. Moore 1993,
A. Chenciner and R. Montgomery 2000

Figure-eight and slalom solutions

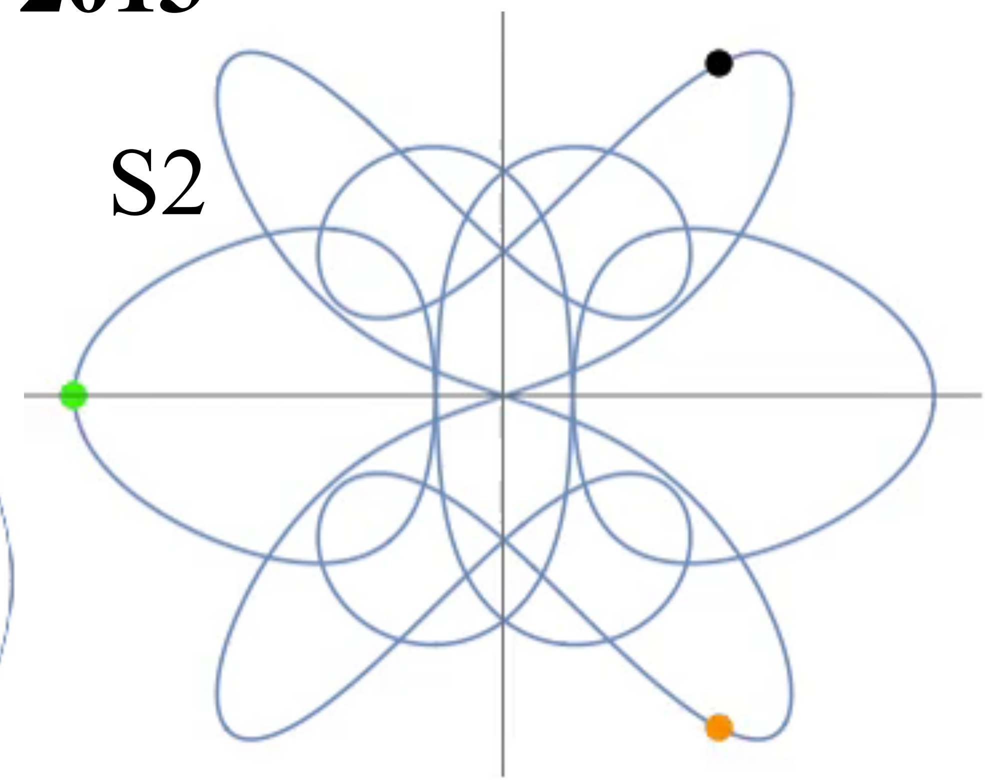
figure-eight: Moore 1993, Chenciner & Montgomery 2000

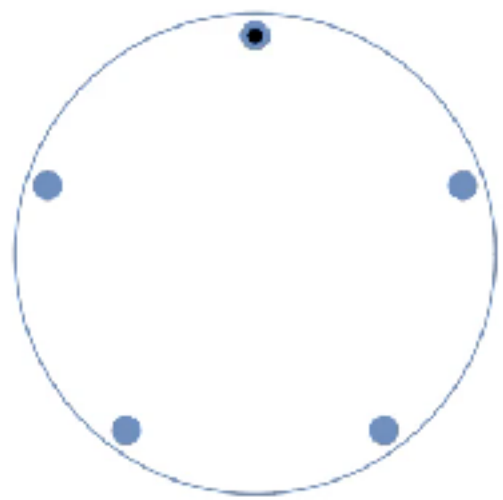
slalome: Šuvakov & Dmitrašinović 2013

S1



S2





$k = 5$ slaloms

S3

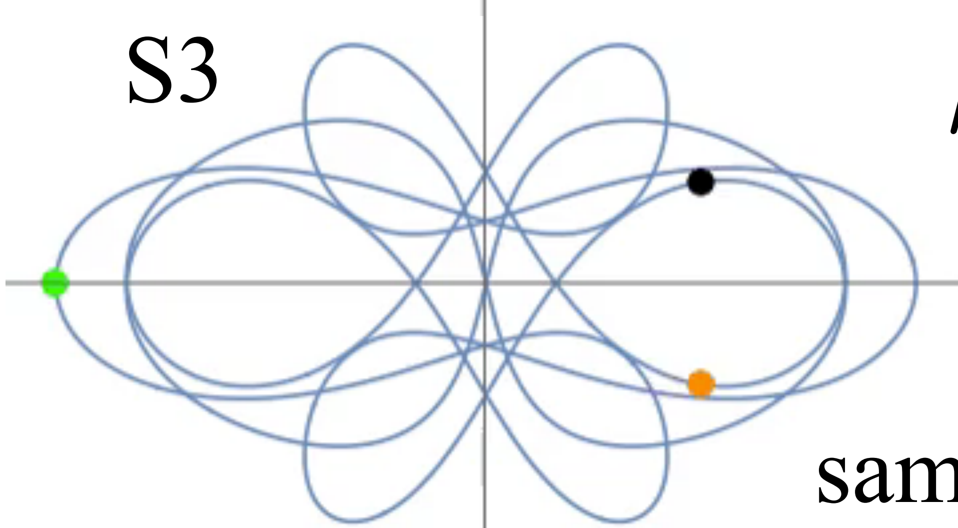
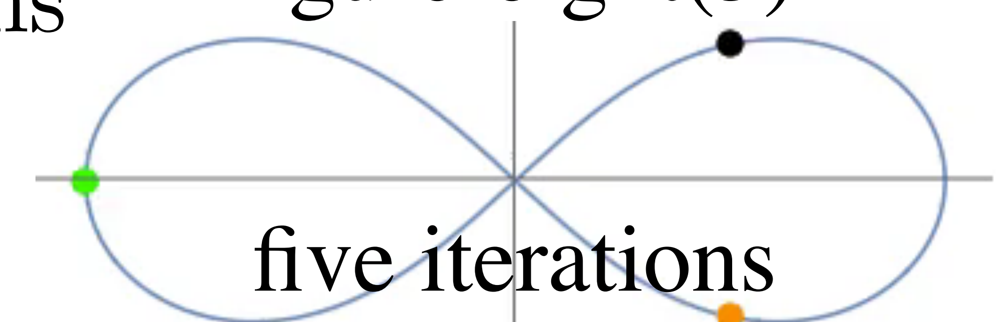


figure-eight(5)



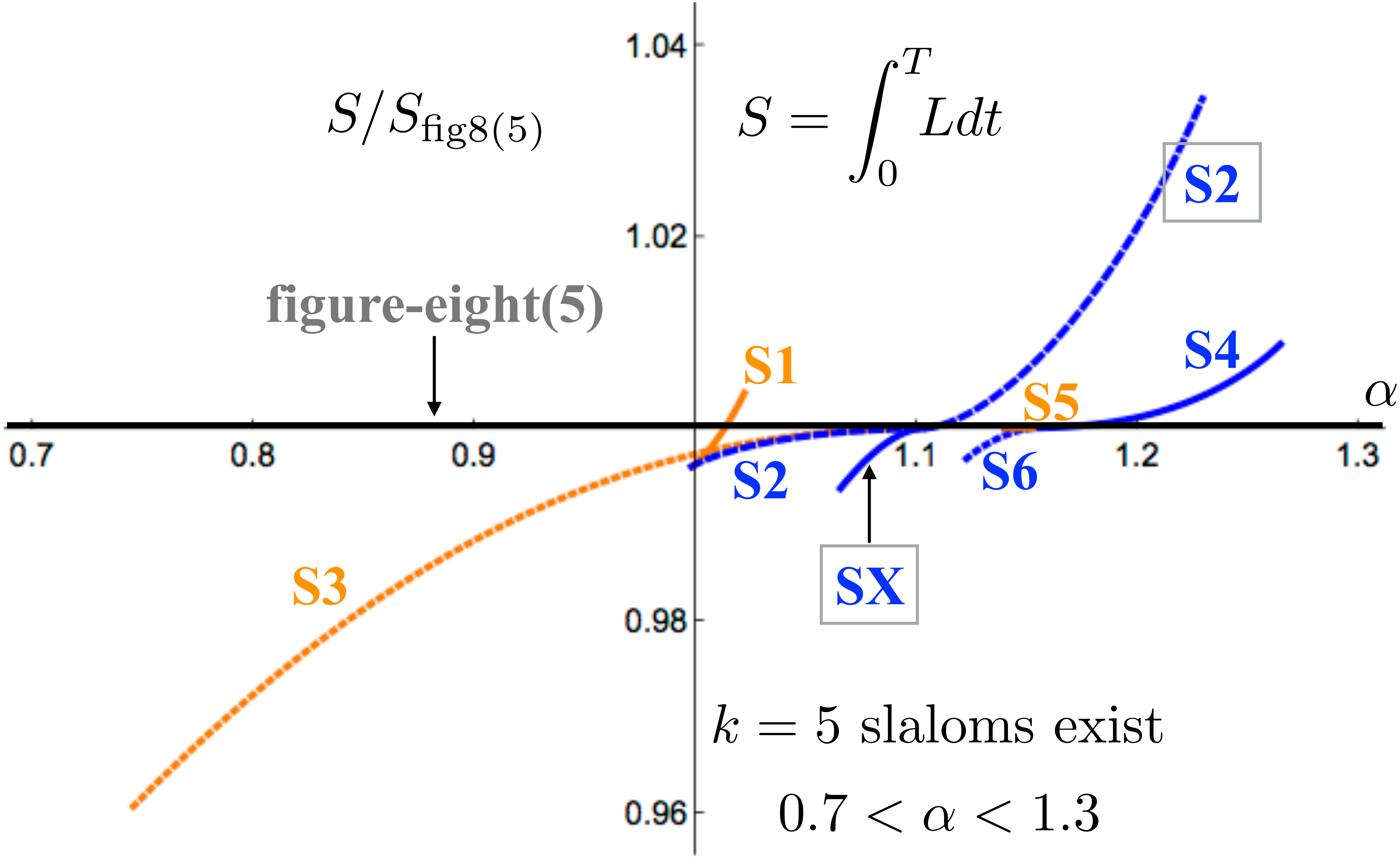
five iterations

same homotopy class

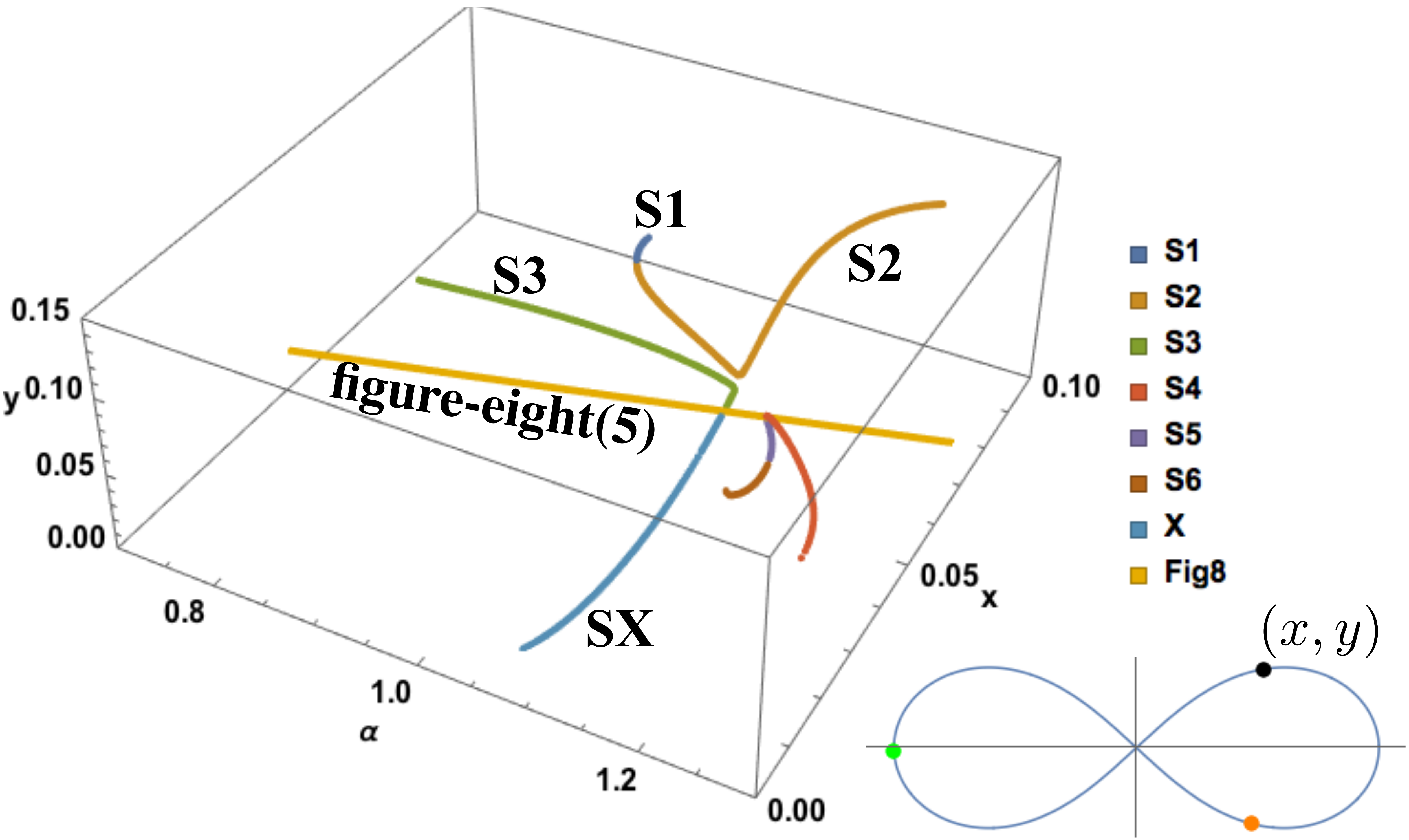
Slalom Solutions

- **Šuvakov M.**
  - **Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere, Celest. Mech. Dyn. Astron. 119, 369–377 (2014)**
- **Šuvakov M., Dmitrašinović V.**
  - **Three classes of Newtonian three-body planar periodic orbits, Phys. Rev. Lett. 110(11), 114301 (2013)**
- **Šuvakov M., Dmitrašinović V.**
  - **A guide to hunting periodic three-body orbits, Am. J. Phys. 82, 609–619 (2014)**
- **Šuvakov M., Shibayama M**
  - **Three topologically nontrivial choreographic motions of three bodies, Celest. Mech. Dyn. Astron. 124, 155–162 (2016)**

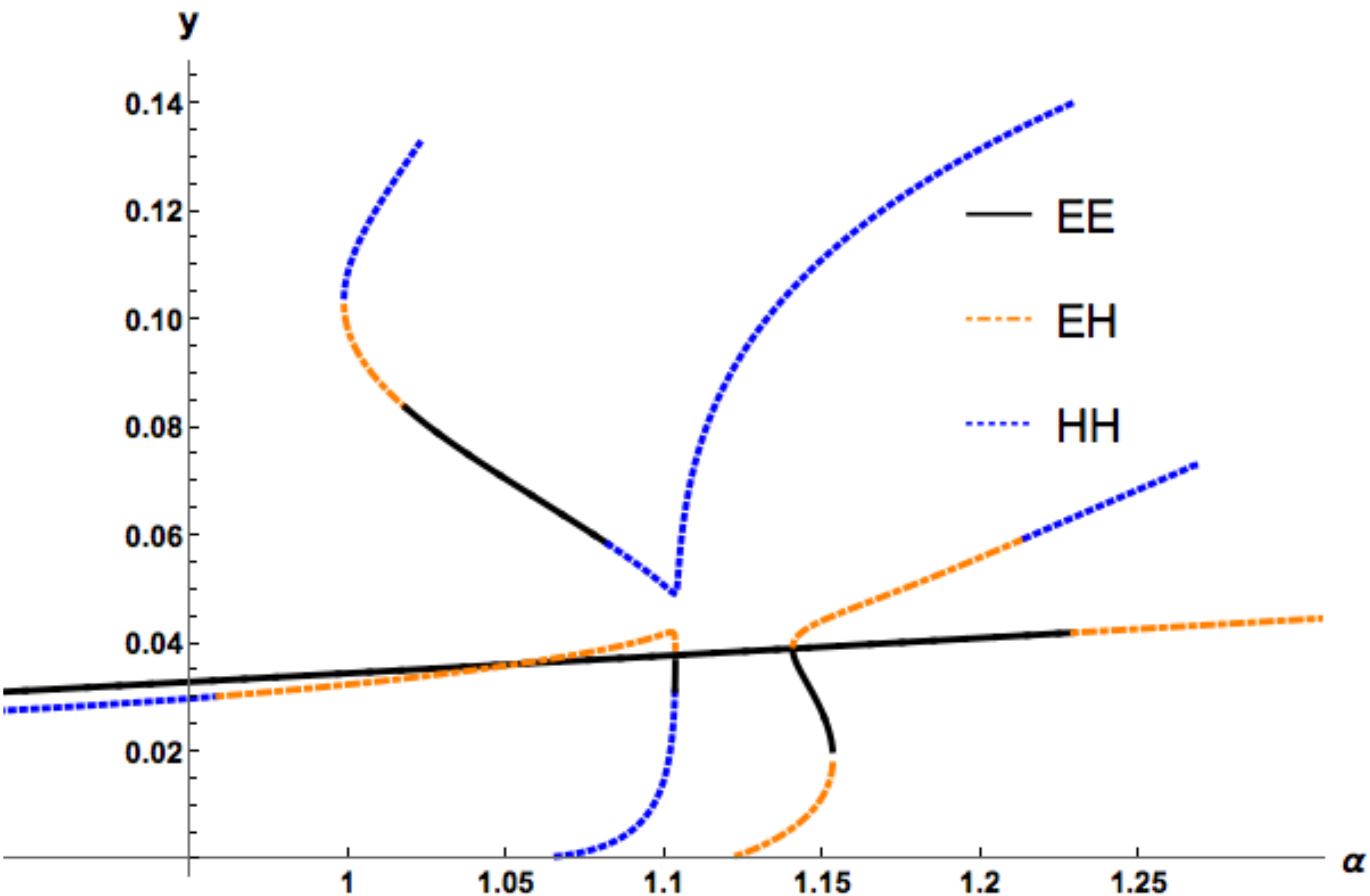
Continuation of solutions



continuation of solutions



Linear stability



Continuation of
known slalom solutions k=5,
S1, S2, S3

↓

Period 5 bifurcations from the figure-eight

Linear stability
Second derivative of action for figure-eight(5)

↓

other bifurcated choreographic solutions?

Yes!

Linear stability, Floquet matrix

small variation $\delta q(t), \delta p(t) \in \mathbb{R}^6$
around a periodic solution $q(t), p(t)$
with $q(t+T) = q(t), p(t+T) = p(t)$

equation of motion for $\Psi(t) = \begin{pmatrix} \delta q(t) \\ \delta p(t) \end{pmatrix}$, $\dfrac{d}{dt}\Psi(t) = B(t)\Psi(t)$

$$B(t) = \begin{pmatrix} 0 & \delta_{ij} \\ \dfrac{\partial^2 V}{\partial q_i \partial q_j} & 0 \end{pmatrix}$$

Green function $G(t)$, $\dfrac{d}{dt}G(t) = B(t)G(t) \Rightarrow \Psi(t) = G(t)\Psi(0)$

Floquet matrix $M = G(T)$

$$M\Psi = \mu\Psi$$

μ: characteristic multiplier

characteristic multipliers μ and linear stability

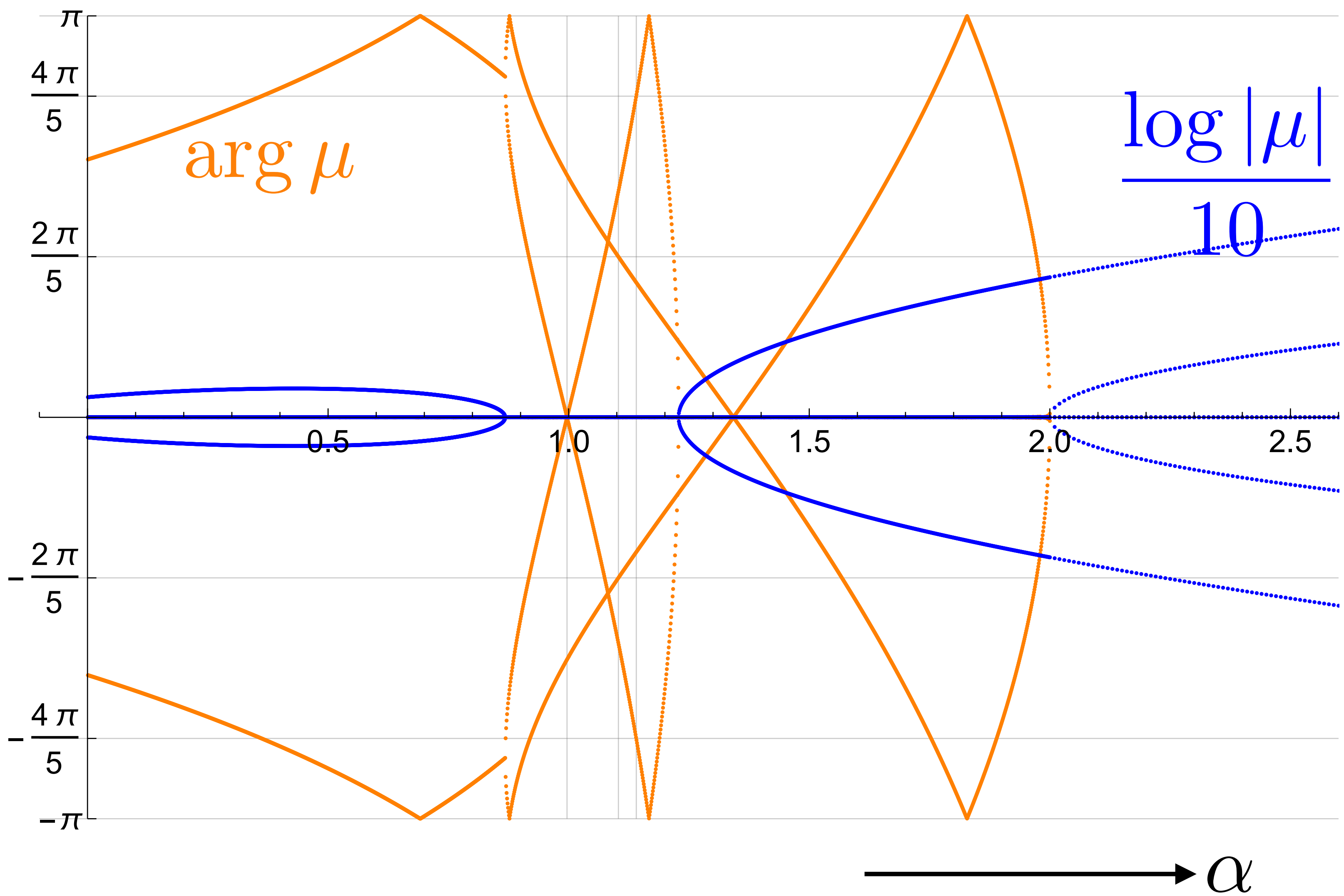
$$M\Psi = \mu\Psi$$

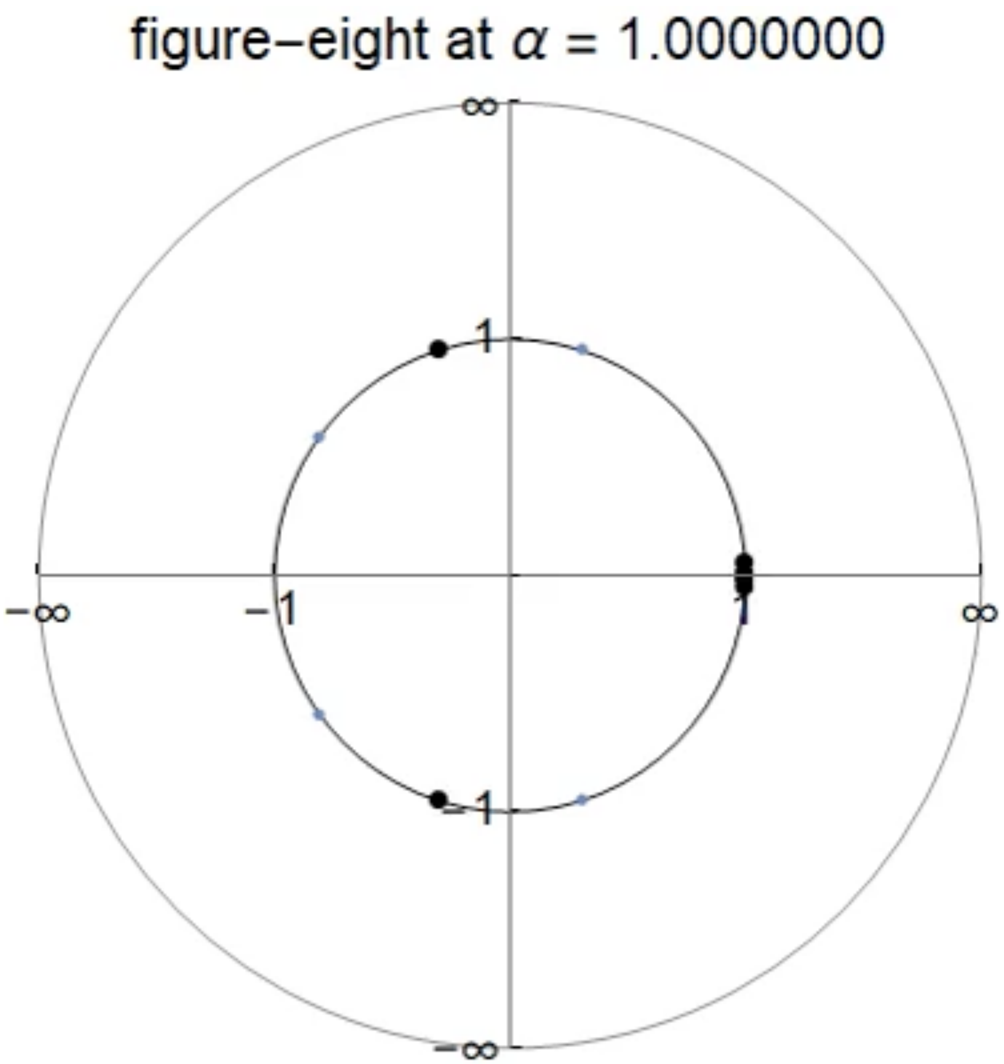
$$8 \text{ trivial } \mu = 1,\; 4 \text{ non-trivial } \mu$$

$$M : \text{symplectic } \Rightarrow \mu, 1/\mu, \mu^*, 1/\mu^*$$

$$\begin{cases} |\mu| = 1 \rightarrow \text{ E: elliptic,} \\ \\ |\mu| \neq 1 \rightarrow \text{ H: hyperbolic} \end{cases}$$

characteristic multipliers for figure-eight(1)



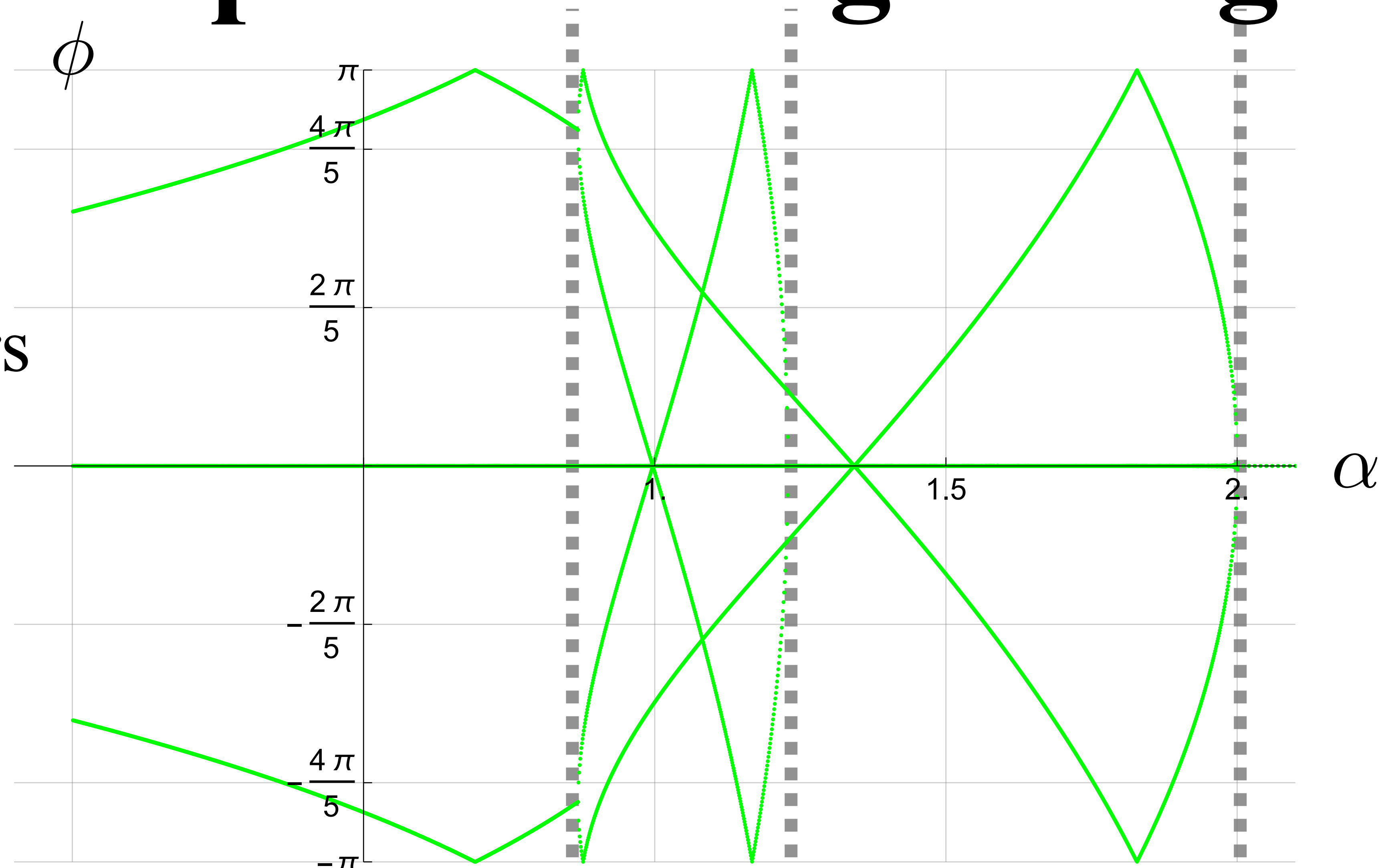


$$\mu_i \in \mathbb{R}, i = 1, 2, 3, 4$$

arguments of characteristic multipliers for figure-eight

4 multipliers

$\mu = \rho e^{i\phi}$



$\rho_k \neq 1$ | $\rho_k = 1$ | $\rho_1 = \rho_2 = 1$, $\rho_3 = 1/\rho_4 \neq 1$ | $\rho_k \neq 1$

HH | EE | EH | HH

Hyperbolic, Elliptic

Hessian of action

Equal mass planar three-body problem

$$L = \frac{1}{2}\sum_{\ell}\left(\frac{dq_\ell}{dt}\right)^2 + U,\ U = \sum_{i\neq j} V(|q_i - q_j|)$$

$$S[q+\delta q] = S[q] + \underbrace{\delta S[q]}_{=0} + \frac{1}{2}\int_0^T dt \sum_{i,j} \delta q_i \underbrace{\left(-\delta_{ij}\frac{d^2}{dt^2} + \frac{\partial^2 U}{\partial q_i \partial q_j}\right)}_{=\mathcal{H}\text{: Hessian}} \delta q_j$$

$$\mathcal{H}\Psi = \lambda\Psi,\ \ \Psi = \begin{pmatrix}\delta q_0\\ \delta q_1\\ \delta q_2\end{pmatrix},\ \ \delta q_\ell = \begin{pmatrix}\delta q_{\ell x}\\ \delta q_{\ell y}\end{pmatrix} \in \mathbb{R}^2.$$

$\delta q_\ell(t+T) = \delta q_\ell(t)$:only impose periodicity for variations

$$T = 5T_{\text{figure-eight}}$$

symmetries

Since the figure-eight solution has symmetries

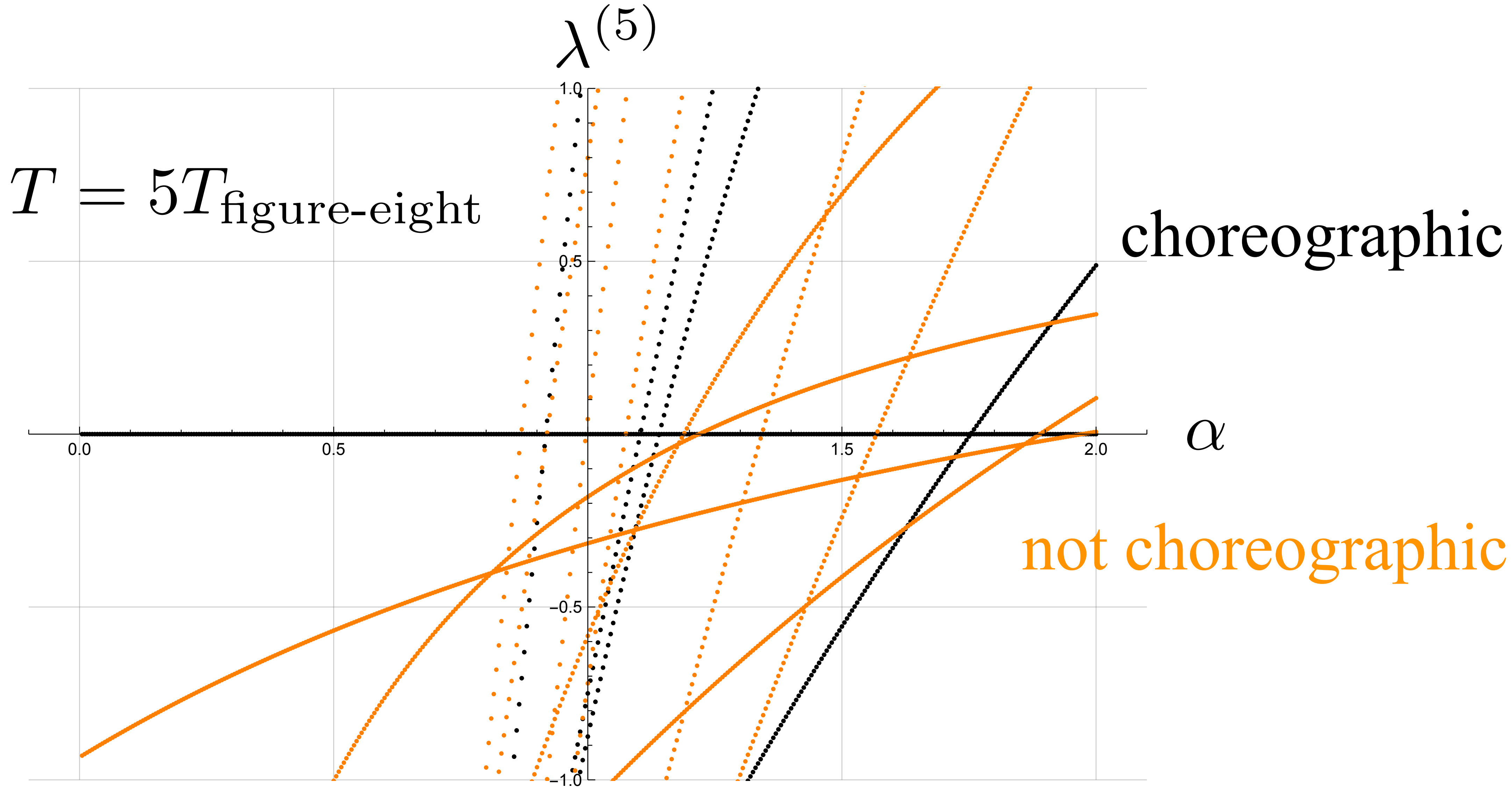
$$\begin{cases} \mathcal{C} & : \text{choreograpfic} \\ \mathcal{M} & : t \to t + T/2 \\ \mathcal{T} & : t \to -t \end{cases}$$

So, the Hessian around the figure-eight is invariant under the symmetries.

Then, the eigenfunction $\mathcal{H}^{(5)}\Psi = \lambda\Psi$

can be classified by $\{\mathcal{C}, \bar{\mathcal{C}}\}, \mathcal{M}^{\pm}, \mathcal{T}^{\pm}$

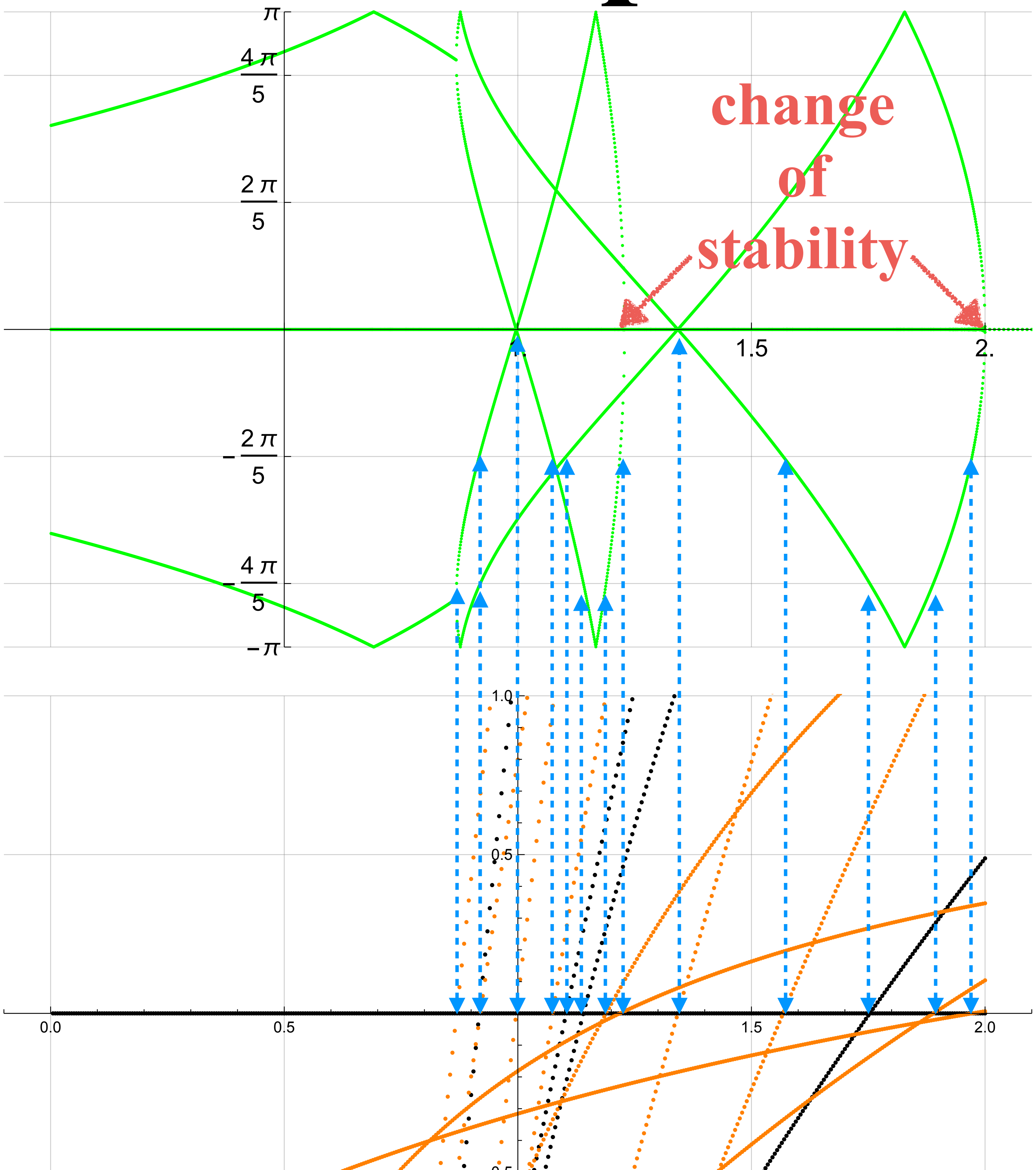
eigenvalues of Hessian of Action



$$\mathcal{H}^{(5)}\Psi = \lambda^{(5)}\Psi$$

every eigenvalues here are **doubly degenerated**

1 to 1 correspondence



$$\phi = \pm \frac{2n\pi}{5}$$

1 to 1
with
2 exceptions

$$\lambda^{(5)} = 0$$

1 to 1

characteristic multipliers: $\mu^5 = 1$

↕ ↘

bifurcation + change of stability

↕

eigenvalues of Hessian of Action: $\lambda^{(5)} = 0$

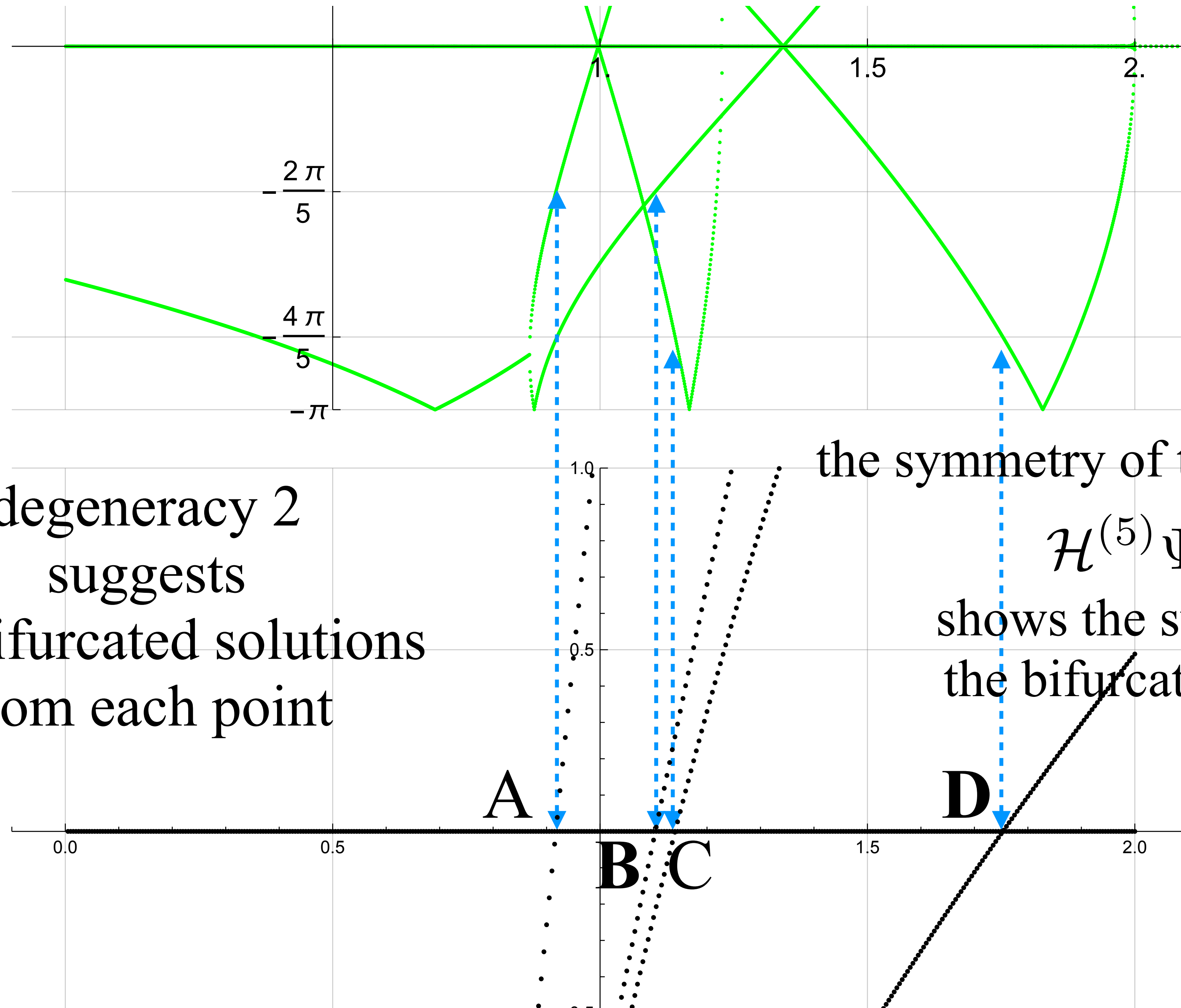
multiplicity of $\lambda^{(5)}$ = "number" of bifurcated solutions

This correspondence is confirmed for figure-eight(1) under Lennard-Jones and homogeneous potential

Question: for figure-eight(5) ?

4 lambda=0 points for choreographic variation

The correspondence is confirmed for B, C, D.



degeneracy 2
suggests
"2" bifurcated solutions
from each point

$\lambda^{(5)} = 0$

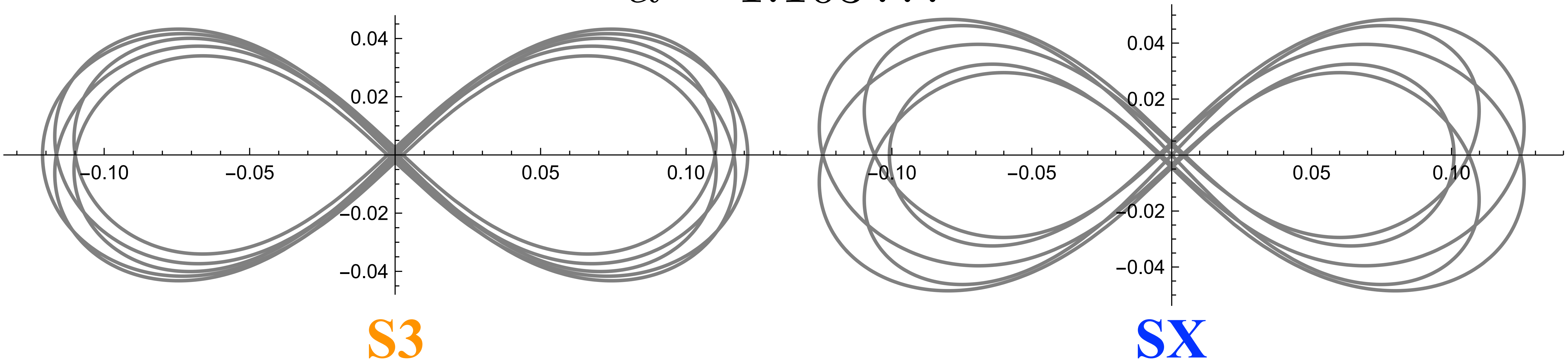
the symmetry of the eigenfunction

$\mathcal{H}^{(5)}\Psi = 0$

shows the symmetry of
the bifurcated solution

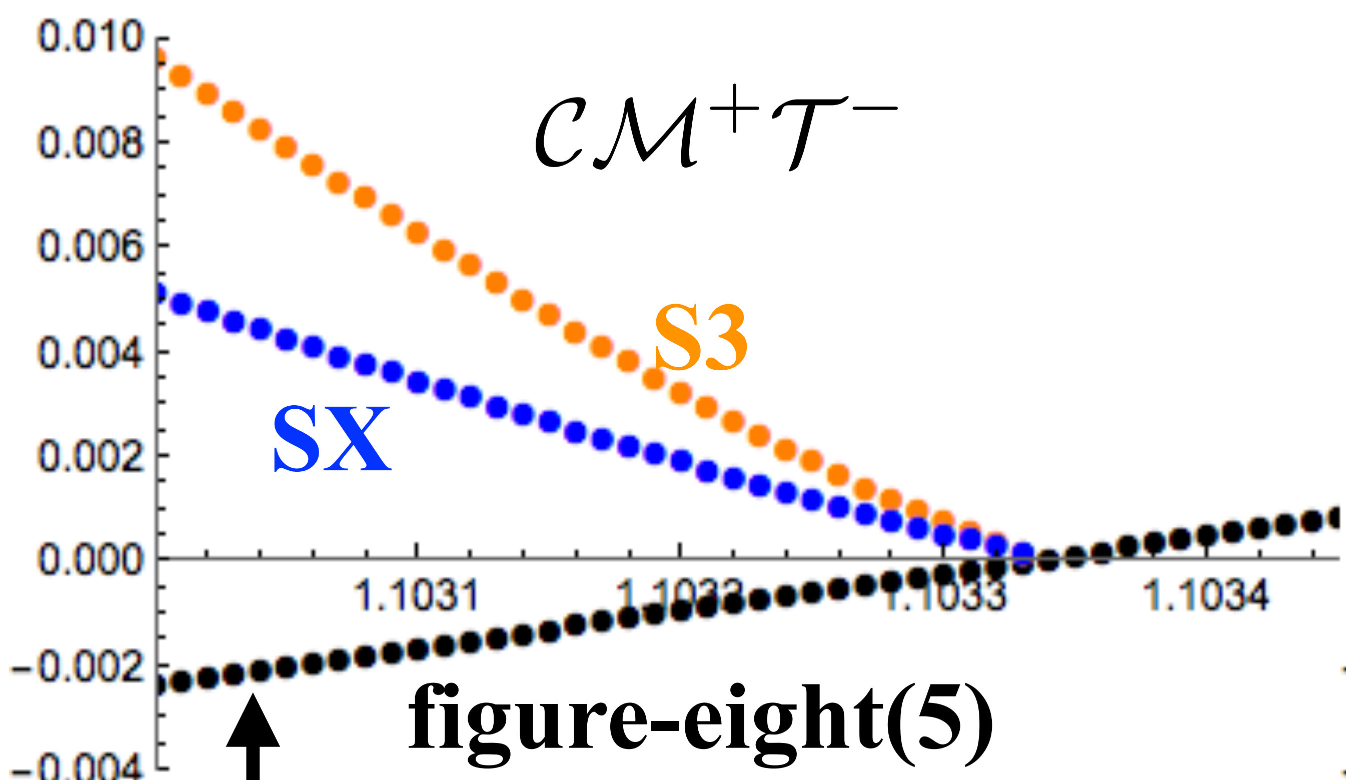
bifurcated solutions near the point B

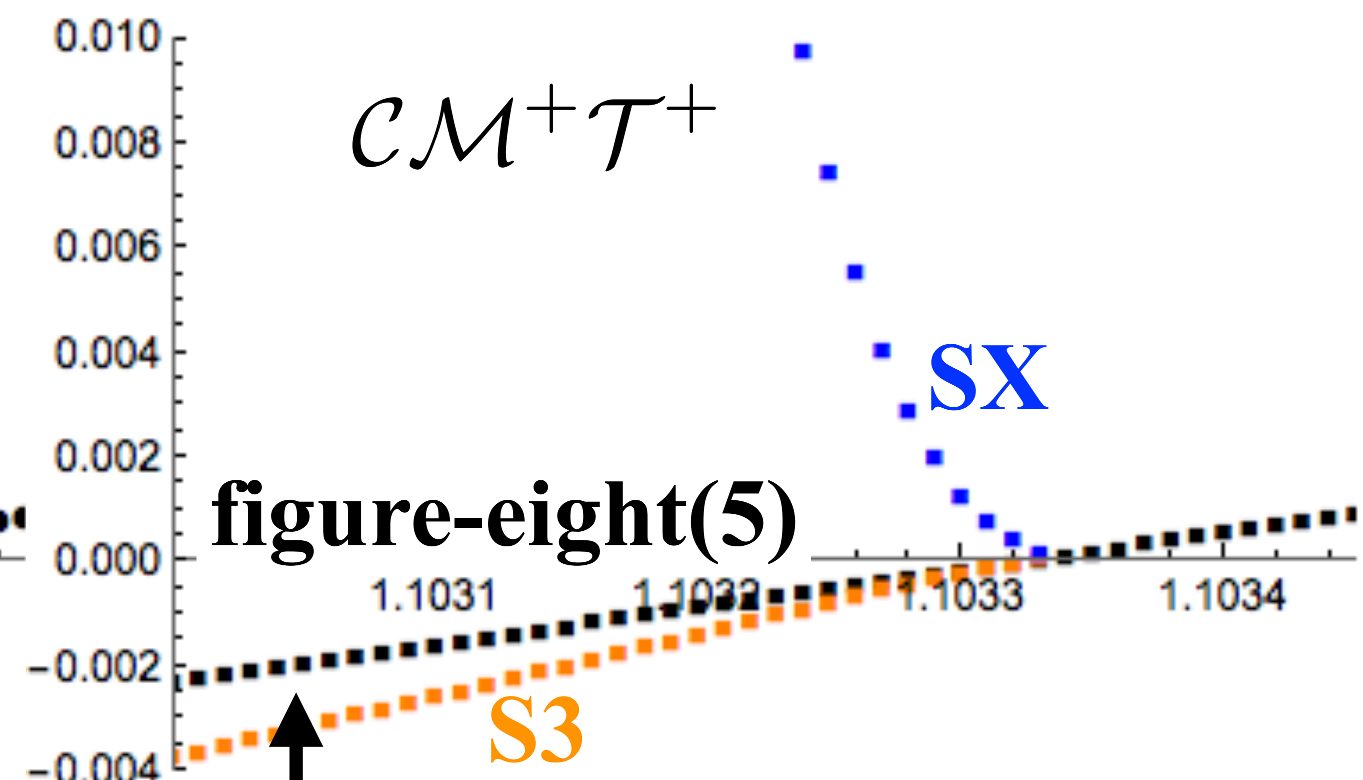
$\alpha \sim 1.103\ldots$



they are choreographic and keep the same symmetry as the figure-eight

Eigenvalues of H near branch point B





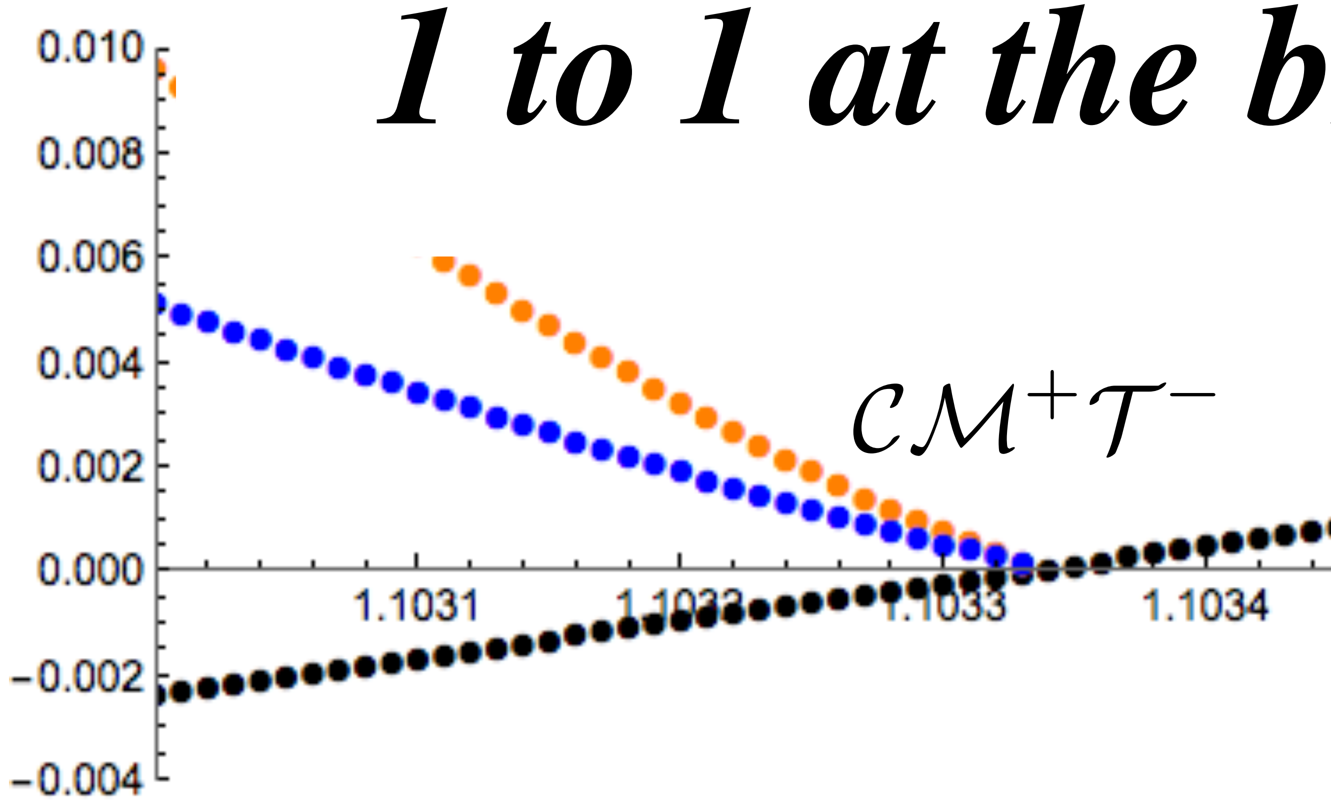
the eigenvalues of figure-eight(5) shown here are degenerated

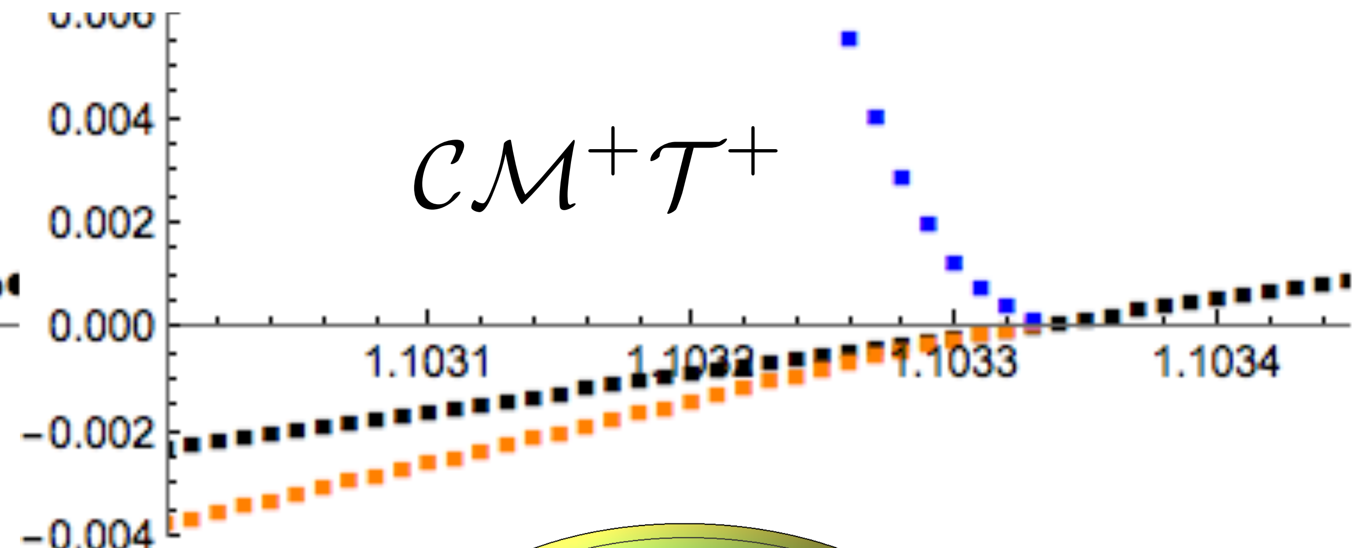
symmetries for δq

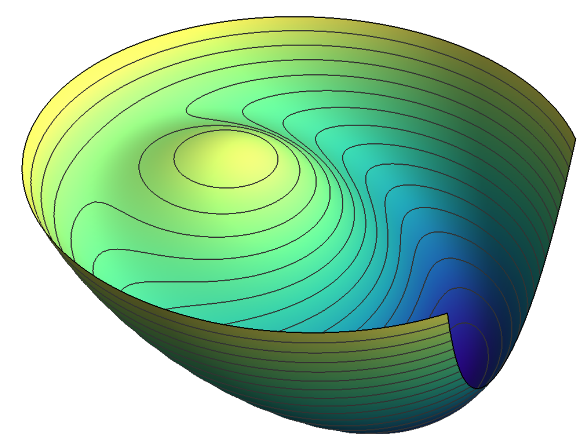
$$\begin{cases} \mathcal{C} & : \text{choreograpfic} \\ \mathcal{M} & : t \to t + T/2 \\ \mathcal{T} & : t \to -t \end{cases}$$

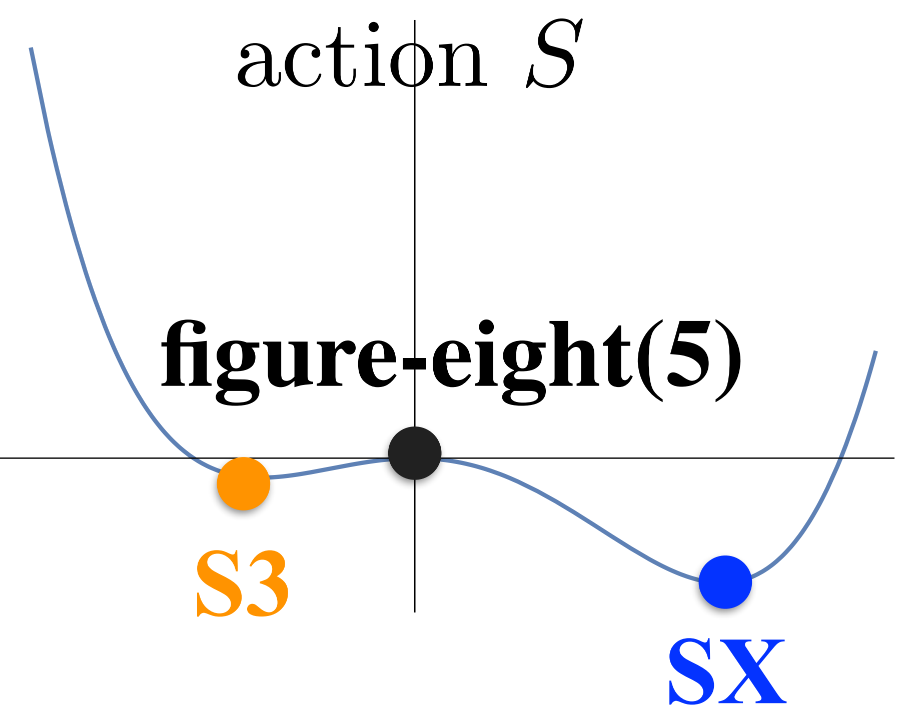
figure-eight q has $\mathcal{C}\mathcal{M}^{+}\mathcal{T}^{-}$

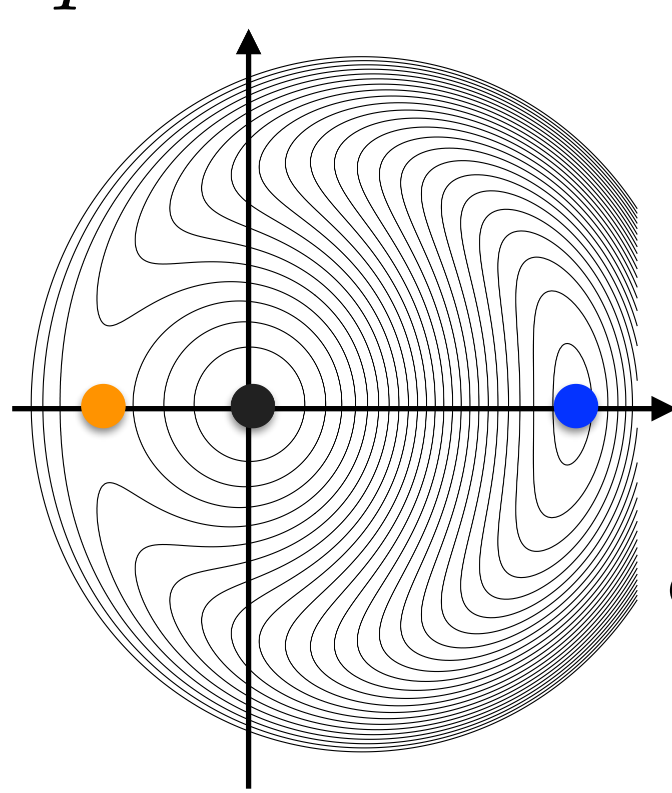
1 to 1 at the branch point B







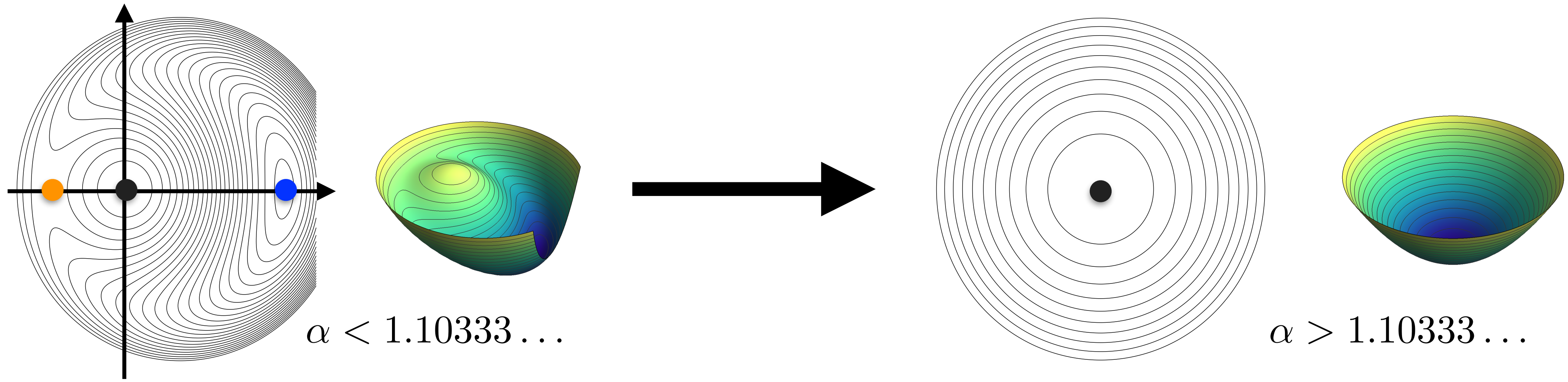




the same symmetry of the figure-eight(5)

→ a single local minimum

Speculation for the action



Images are just a speculation for the action based on the behaviour of eigenvalues of Hessian of the action shown in the previous slide. We don't calculate the action directly. The XY directions represent the two eigenfunction of the Hessian for figure-eight(5),

$$\mathcal{H}^{(5)} e_k = \lambda^{(5)} e_k, k = 1, 2.$$

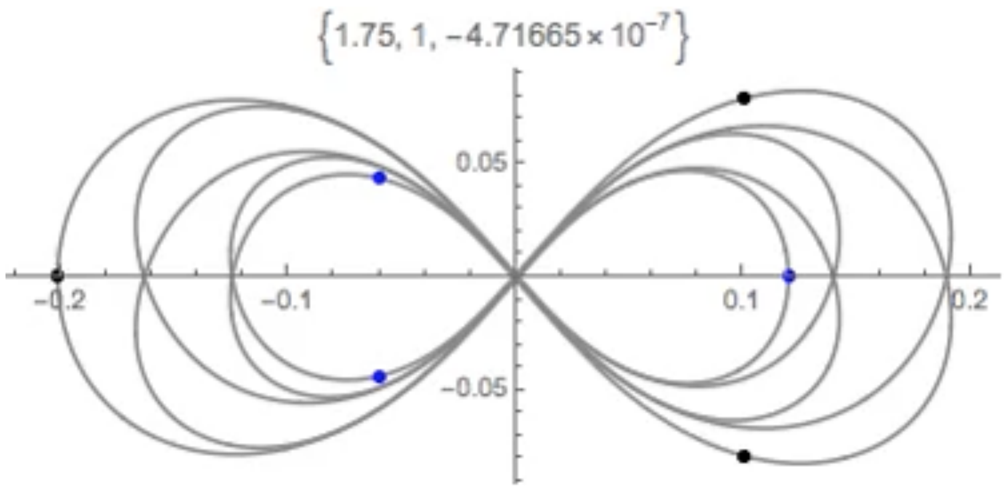
Since the eigenvalues for the figure-eight that we are noticing is doubly degenerated, we have two eigenfunctions e_1 and e_2 that are represented by δq and $\delta \tilde{q}$ in the previous slide.

For $\alpha < 1.10333\ldots$, the eigenvalues for figure-eight(5) are (both) negative, the eigenvalues for S_3 are positive and negative for each direction (see the previous slide) and the eigenvalues for S_X are both positive. Therefore, the point figure-eight is a local maximum, S_3 is a saddle, and point S_X is a local minimum.

On the other hand, for $\alpha > 1.10333\ldots$, the solutions S_3 and S_X vanish, and only the figure-eight(5) has (doubly degenerated) positive eigenvalues. Therefore, the action has simple local minimum at the figure-eight.

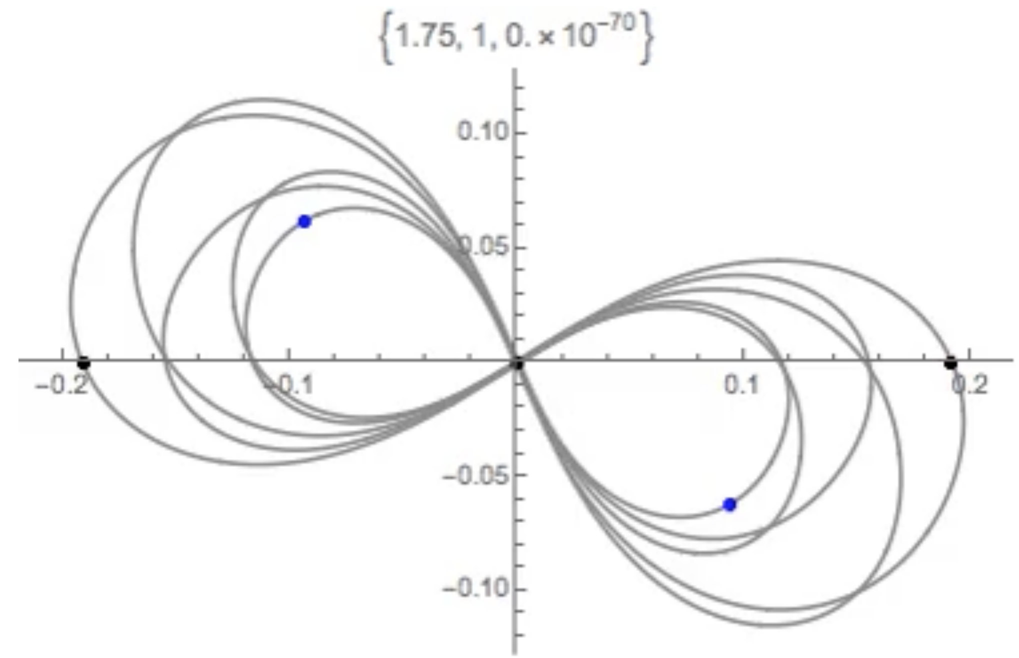
near the point D

$$\alpha \sim 1.75\ldots$$



$c \neq 0$

X-axis symmetric,
not Y-axis



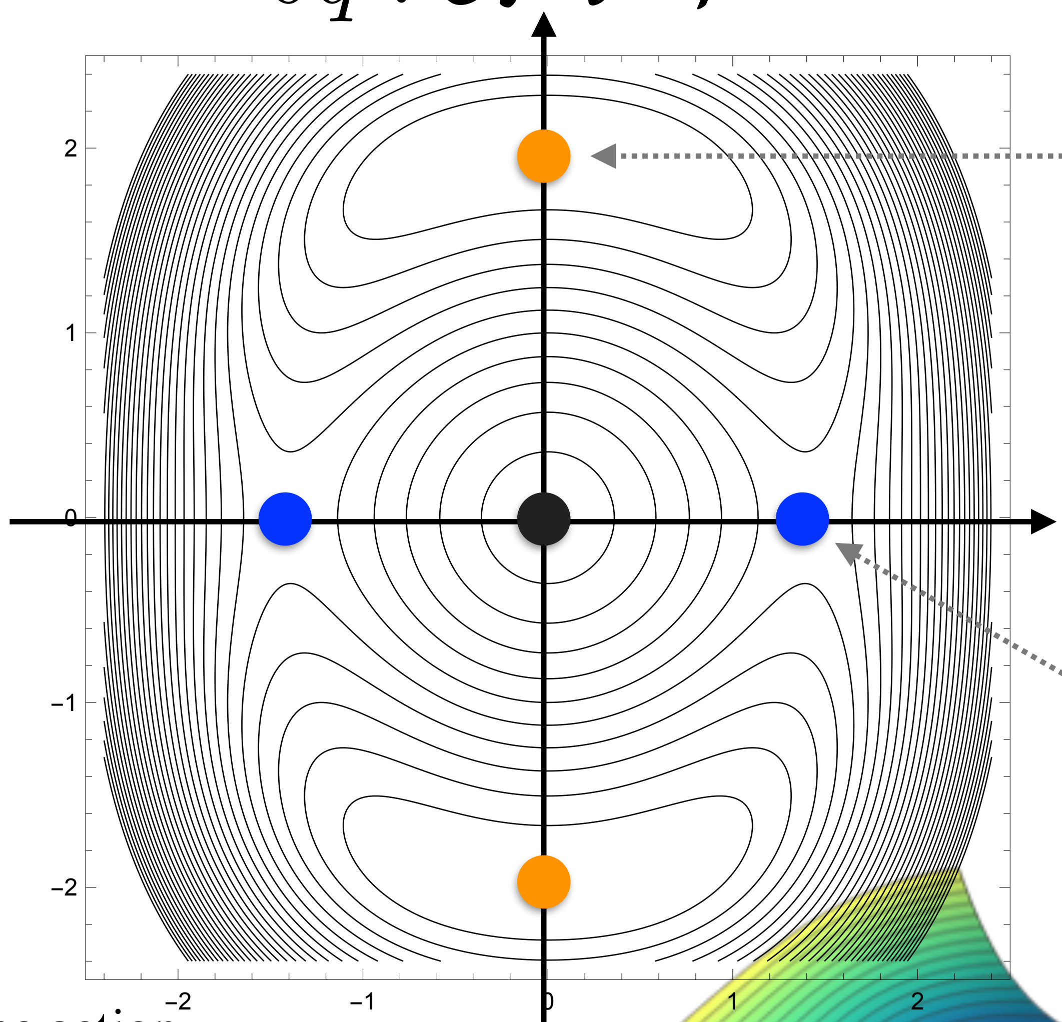
$c = 0$

Origin symmetric

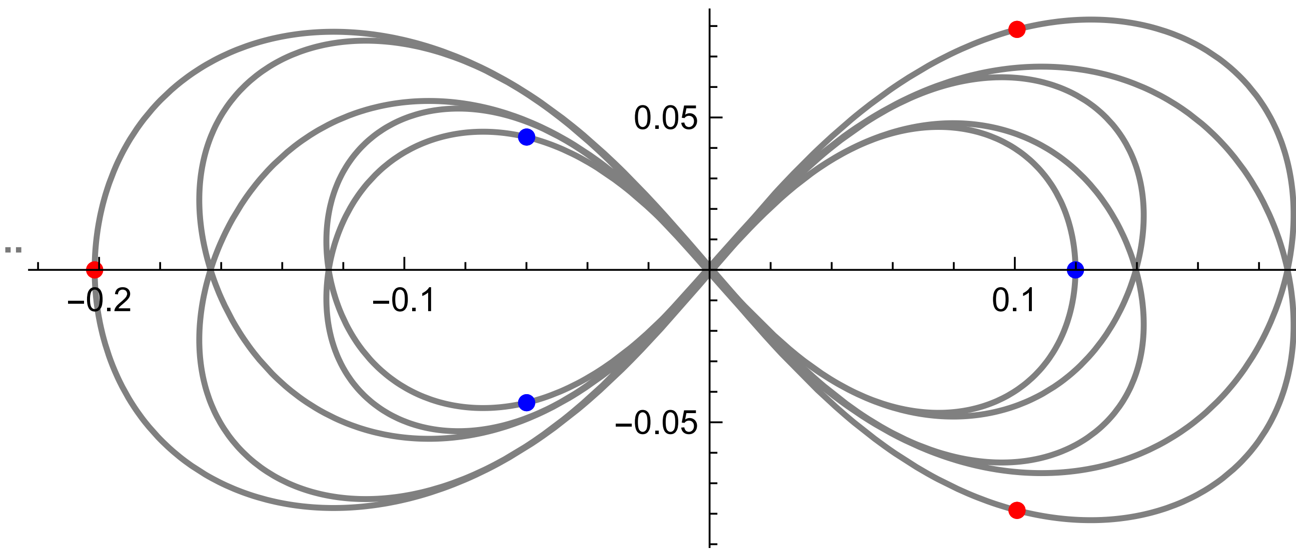
choreographic
but the figure-eight symmetry is broken

1 to 1 at the branch point D

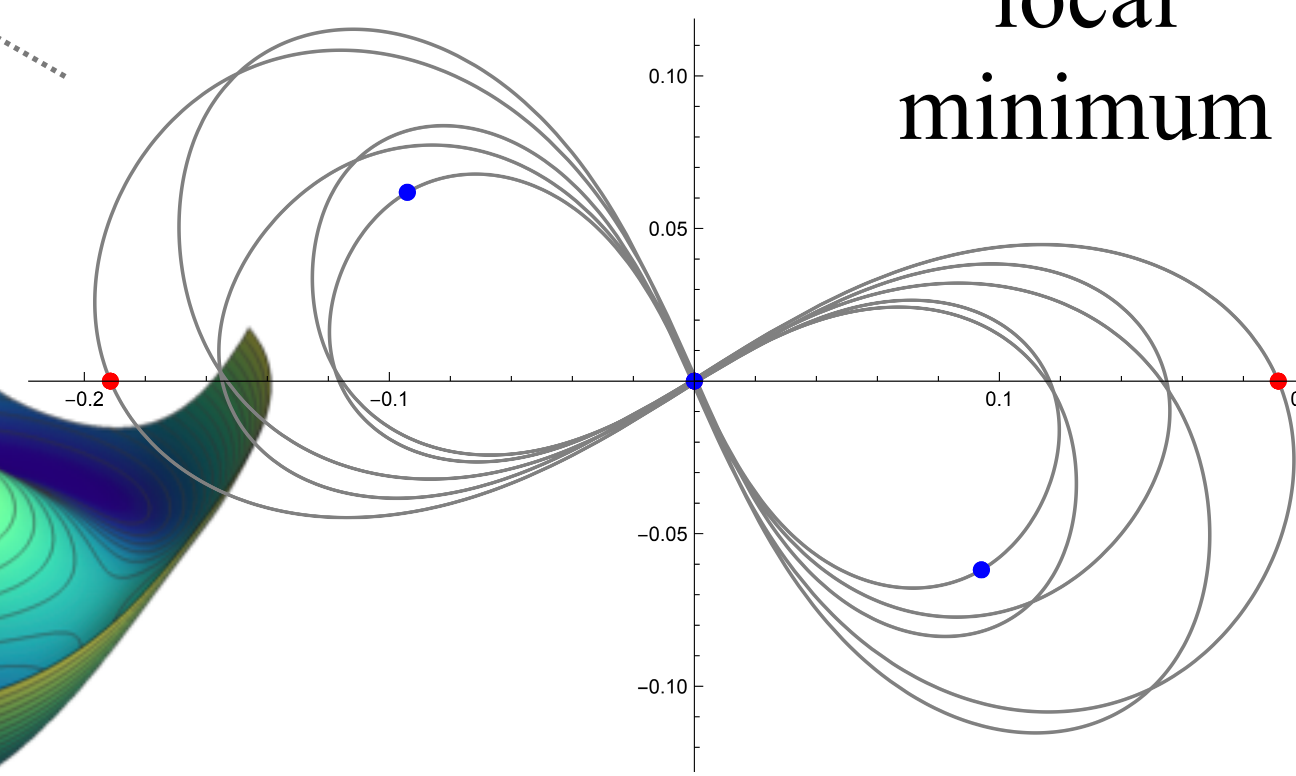
$\delta\tilde{q} : \mathcal{C}\mathcal{M}^{-}\mathcal{T}^{+}$

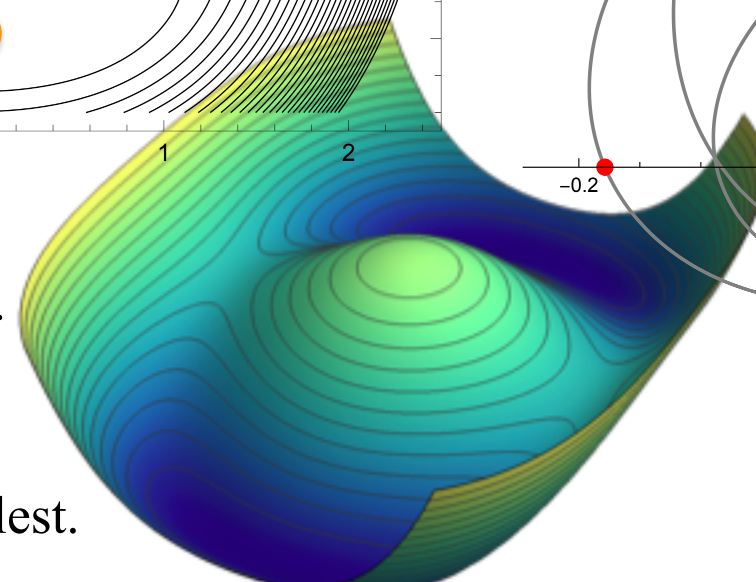


$\delta q : \mathcal{C}\mathcal{M}^{-}\mathcal{T}^{-}$



→ a single local minimum





Images for the action,
which are just a speculation.
We didn't calculate the eigenvalues yet.

We calculated the action.
The figure-eight has largest action
and X-symmetric solution has the smallest.

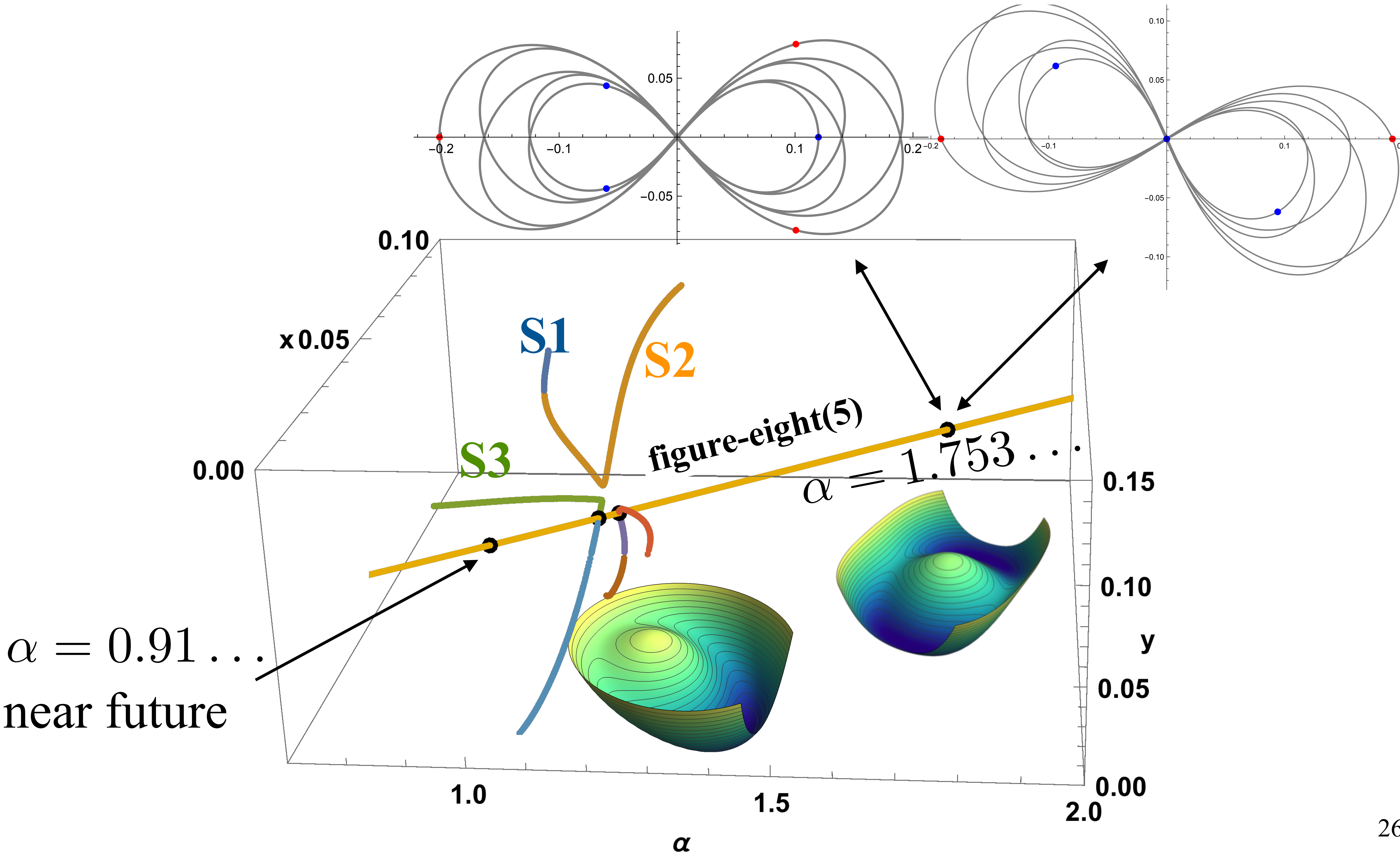
summary

not "proved"

We observed the following properties for the bifurcations from figure-eight(5);

- **characteristic multiplier μ=1 → bifurcation or change of stability**
- **eigenvalues of Hessian λ=0, $\mathcal{H}^{(5)}\Psi = 0 \rightleftarrows$ bifurcation (1 to 1)**
- **multiplicity of λ=0 $\rightleftarrows$ "number" of bifurcated solutions**
- **symmetry of the eigenfunction δq predict the symmetry of bifurcated solution q+δq**
- **eigenvalues of Hessian for the figure-eight(5) and the bifurcated solutions suggest the behaviour of action in function space**

Summary



S1
S2
S3
figure-eight(5)
$\alpha = 1.753\ldots$
$\alpha = 0.91\ldots$
near future
x
y
α
0.10
0.05
0.00
0.15
1.0
1.5
2.0