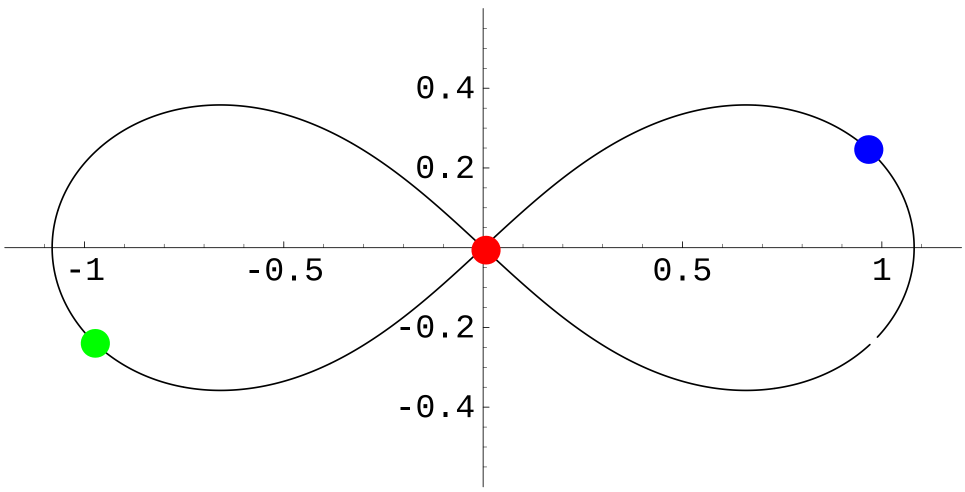
Proof of Non-Conservation of the Moment of Inertia of Three-Body Choreography



— Inconstancy of I —

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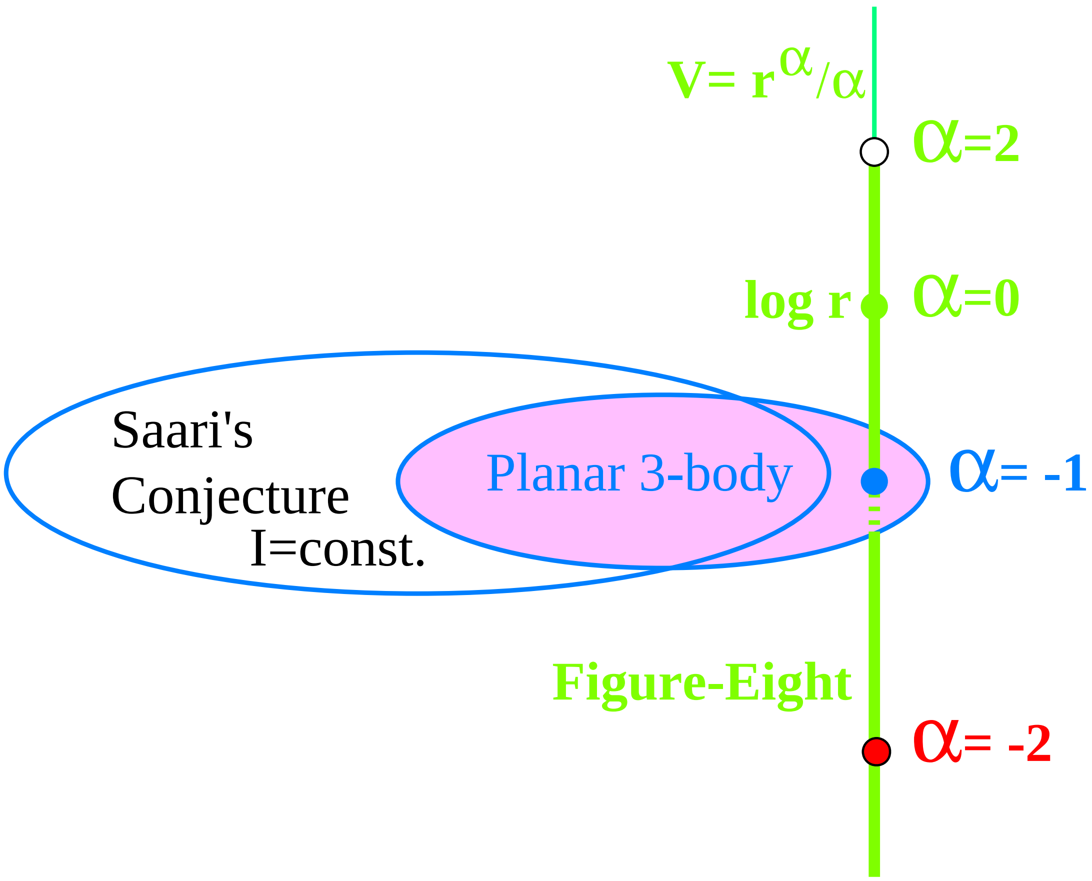
Hasselt, Belgium

$$I = \frac{1}{2}\sum_i \mathbf{r}_i^2, \quad V_\alpha = \begin{cases} r^\alpha/\alpha & \text{for } \alpha \neq 0 \\ \log r & \text{for } \alpha = 0 \end{cases}$$



Chenciner's Problem:

Figure-Eight

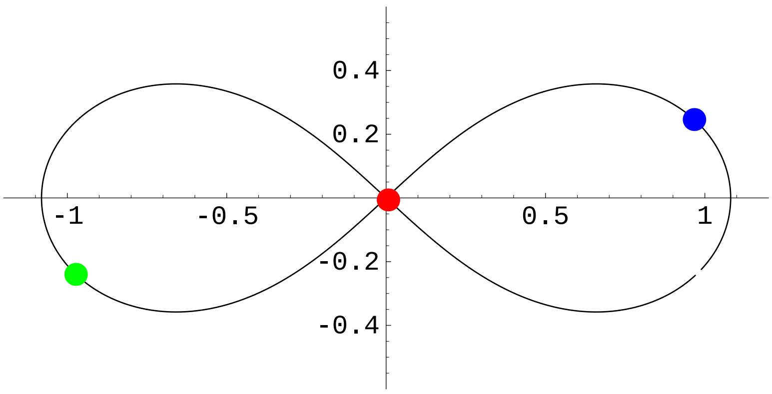
I =const. only when $\alpha = -2$.

A. Chenciner, 2002. Proc. Conf. on Non-linear functional analysis (Taiyuan) (World Scientific)

What is the "Figure-Eight"?

OR

What properties of the Eight
make the problem easy?



$\Downarrow$

Motions with conditions

Equal masses

(I) I =const. $I \neq 0$,

(II) $\mathbf{L} = \sum \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{0}$

and

(III) $\mathbf{x}_3(0) = \mathbf{0}$ (Center of Mass) .

$\Rightarrow$ **Theorem:** NO motion satisfies conditions (I),(II) and (III), except for $\alpha = -2, 2, 4$.

Motions with $\alpha = 2, 4$ are not "Eight".

$\Rightarrow$ Solution of the Chenciner's Problem

Method:

$$\left.\begin{array}{l}\mathbf{x}_3(0) = 0 \\ \mathbf{L} = \sum_i \mathbf{x}_i \times \dot{\mathbf{x}}_i = 0 \\ \frac{dI}{dt}(0) = 0\end{array}\right\} \Rightarrow \text{ initial value: } u, \theta$$

$$\frac{d^2I}{dt^2}(0) \text{ Lagrange-Jacobi identity } \Rightarrow \text{ fix: } u$$

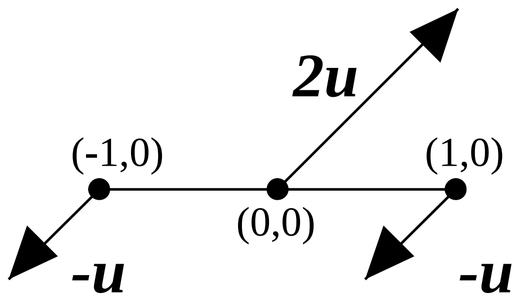
$$\frac{d^4I}{dt^4}(0) \Rightarrow \text{ fix: } \theta$$

$$\frac{d^6I}{dt^6}(0) = 0, \frac{d^8I}{dt^8}(0) = 0, \cdots \Rightarrow \text{ conditions for } \alpha$$

1st derivative:

$$\frac{dI}{dt} = 0, \mathbf{L} = \sum \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{0} \text{ and } \mathbf{r}_3(0) = \mathbf{0}.$$

$$\Rightarrow \begin{cases} \mathbf{r}_3(0) = \mathbf{0}, \\ \mathbf{r}_1(0) = -\mathbf{r}_2(0), \\ \dot{\mathbf{r}}_1(0) = \dot{\mathbf{r}}_2(0) = -\mathbf{u}, \\ \dot{\mathbf{r}}_3(0) = 2\mathbf{u} \end{cases}$$



Simó's initial value for E and H3 orbit.

Parameters: $\mathbf{u} = u(\cos\theta, \sin\theta)$.

—— details ——

$$\mathbf{r}_3(0) = \mathbf{0} \Rightarrow \mathbf{r}_1(0) = -\mathbf{r}_2(0),$$

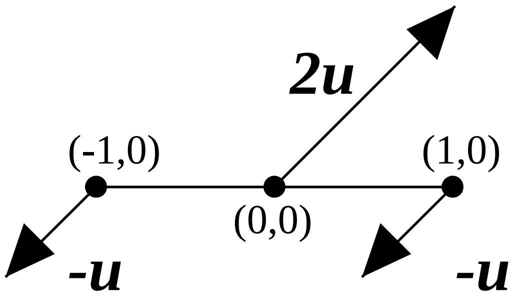
$$\begin{cases} \frac{dI}{dt} = 0 & \Rightarrow & \mathbf{r}_1(0) \cdot (\dot{\mathbf{r}}_1(0) - \dot{\mathbf{r}}_2(0)) = 0 \\ \mathbf{L} = \mathbf{0} & \Rightarrow & \mathbf{r}_1(0) \times (\dot{\mathbf{r}}_1(0) - \dot{\mathbf{r}}_2(0)) = \mathbf{0} \end{cases}$$
$$\Rightarrow \dot{\mathbf{r}}_1(0) = \dot{\mathbf{r}}_2(0).$$

2nd derivative:

$$\frac{d^2 I}{dt^2}(0) = 0$$

Lagrange-Jacobi identity

$$\Rightarrow 2K = \sum \mathbf{r}_{ij}^{\alpha} \text{ for all } \alpha$$



$$\Rightarrow 6\mathbf{u}^2 = 2^{\alpha} + 2$$

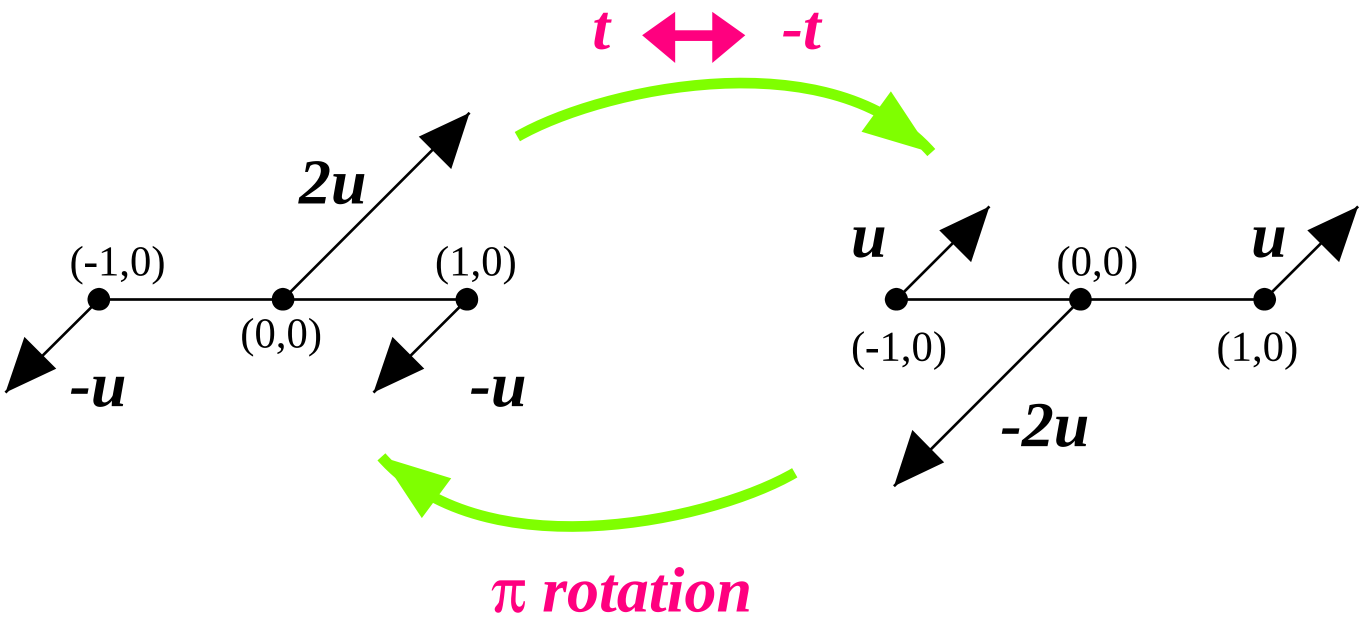
u : fixed for all α

———detail———

$$I = \frac{1}{2}\sum \mathbf{r}_i^2,\ K = \frac{1}{2}\sum \dot{\mathbf{r}}_i^2$$

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\left(\frac{1}{2}\sum \mathbf{r}_i^2\right) = \sum \dot{\mathbf{r}}_i^2 - \sum \mathbf{r}_i \frac{\partial}{\partial \mathbf{r}_i} V_\alpha \\ &= \begin{cases} 2K - \alpha V_\alpha & = 2E - (2+\alpha)V_\alpha \quad \text{for } \alpha \neq 0, \\ 2K - 3 & = 2E - 3 - 2V_0 \quad \text{for } \alpha = 0 \end{cases} \end{aligned}$$

3rd, 5th, 7th, ... derivatives:



$$V_\alpha(\mathbf{r}(-t)) = V_\alpha(\mathbf{r}(t))$$

$$\Rightarrow \frac{d^{2m+1}V_\alpha}{dt^{2m+1}}(0) = 0 \text{ for } m = 1, 2, 3, \cdots$$

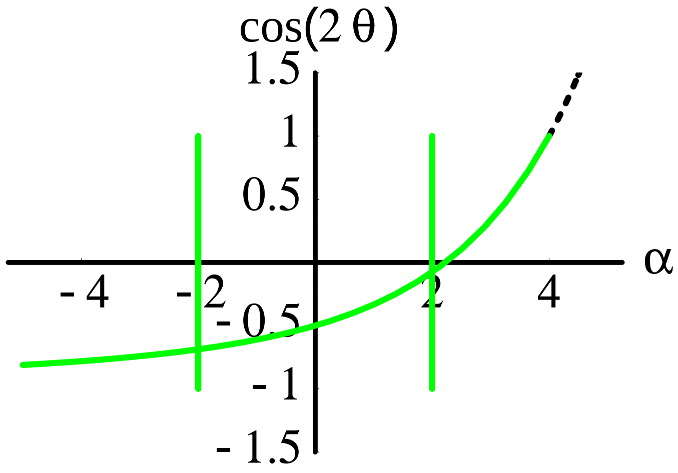
4th derivative:

$$\frac{d^4 I}{dt^4}(0) = -(2+\alpha)\frac{d^2 V_\alpha}{dt^2}(0) = 0$$

$$\frac{d}{dt} = \sum_i \dot{\mathbf{r}}_i \frac{\partial}{\partial \mathbf{r}_i} - \sum_i \frac{\partial V_\alpha}{\partial \mathbf{r}_i}\frac{\partial}{\partial \dot{\mathbf{r}}_i}$$

$$\frac{d^2 V_\alpha}{dt^2}(0) = \frac{(2+2^\alpha)}{2}\Big(3(\alpha-2)\cos(2\theta) - (2+2^\alpha-3\alpha)\Big)$$

$$\Rightarrow \begin{cases} \alpha = -2, 2 \text{ or} \\ \cos(2\theta) = \frac{2^\alpha - 2^2}{2(\alpha-2)} - 1 \end{cases}$$



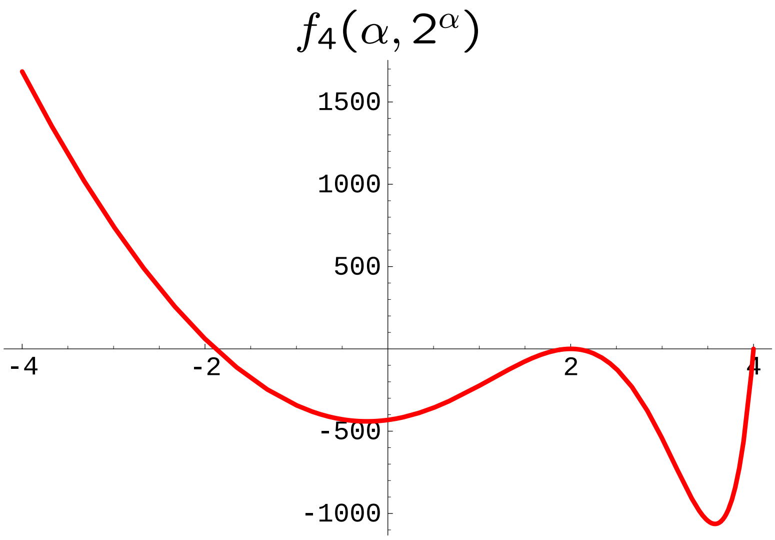
$$\begin{cases} \text{For } \alpha > 4\text{: NO } \theta \\ \text{For } \alpha = -2, 2\text{: Any } \theta \\ \text{Else: } \theta \text{ fixed} \end{cases}$$

6th derivative:

$$\frac{d^6 I}{dt^6}(0) = -(2+\alpha)\frac{d^4 V_\alpha}{dt^4}(0) = 0$$

$$\frac{d^4 V_\alpha}{dt^4}(0) = \frac{(2^\alpha + 2)}{8(\alpha - 2)} f_4(\alpha, 2^\alpha)$$

$$\begin{aligned} f_4(x,y) \quad = \quad & x^2(128 - 36y + 24y^2 + y^3) \\ & -2xy(-112 + 62y + 5y^2) \\ & +8(-32 - 38y + 13y^2 + 3y^3) \end{aligned}$$



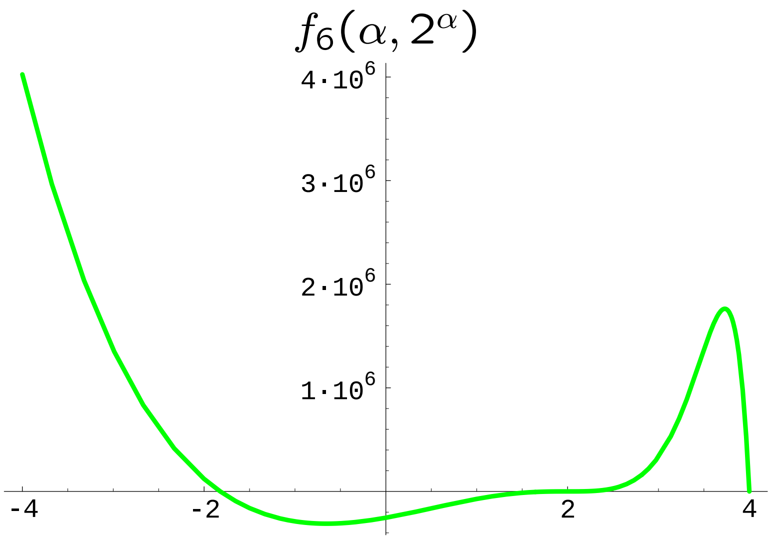
$$\frac{d^6 I}{dt^6}(0) = 0 \quad \Rightarrow \quad \alpha = -2, -1.88599..., 2, 4$$

8th derivative:

$$\frac{d^8 I}{dt^8}(0) = -(2+\alpha)\frac{d^6 V_\alpha}{dt^6}(0) = 0$$

$$\frac{d^6 V_\alpha}{dt^6}(0) = \frac{(2^\alpha + 2)}{32(\alpha - 2)^2} f_6(\alpha, 2^\alpha)$$

$$\begin{aligned} f_6(x,y) \quad = \quad & x^4(6144 + 6496y - 1816y^2 + 60y^3 + 50y^4 + y^5) \\ & -4x^3(10496 + 6520y - 3676y^2 + 508y^3 + 266y^4 + 7y^5) \\ & +4x^2(256 - 10288y - 15032y^2 + 1952y^3 + 1846y^4 + 71y^5) \\ & -16x(-5120 - 10840y - 9428y^2 - 148y^3 + 1186y^4 + 77y^5) \\ & +64(-448 - 1596y - 1860y^2 - 299y^3 + 204y^4 + 30y^5). \end{aligned}$$



$$\frac{d^8 I}{dt^8}(0) = 0 \quad \Rightarrow \quad \alpha = -2, -1.82240..., 2, 4$$

6th and 8th derivative:

$$\frac{d^6 I}{dt^6}(0) = -\frac{(2+\alpha)(2^\alpha+2)}{8(\alpha-2)} f_4(\alpha, 2^\alpha) = 0$$

$$\frac{d^8 I}{dt^8}(0) = -\frac{(2+\alpha)(2^\alpha+2)}{32(\alpha-2)^2} f_6(\alpha, 2^\alpha) = 0$$

$$\begin{aligned} f_4(x,y) \quad = \quad & x^2(128 - 36y + 24y^2 + y^3) \\ & -2xy(-112 + 62y + 5y^2) \\ & +8(-32 - 38y + 13y^2 + 3y^3) \end{aligned}$$

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$$f_4(\alpha, 2^\alpha) = 0 \quad \Rightarrow -1.88599..., 2, 4$$
$$f_6(\alpha, 2^\alpha) = 0 \quad \Rightarrow -1.82240..., 2, 4$$

$$\Downarrow$$

Proof

Problem:

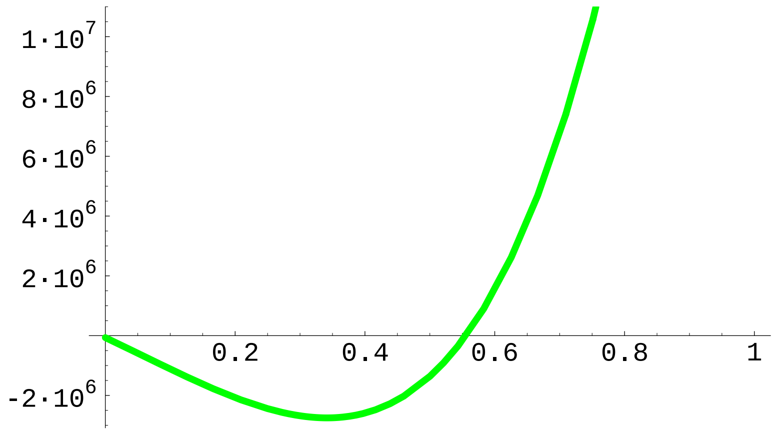
$f_4(\alpha, 2^\alpha)$, $f_6(\alpha, 2^\alpha)$ are polynomial of α and 2^α.

Resultant $R(y)$: Polynomial of $y = 2^{\alpha}$.

$$L(x,y)f_6(x,y) - M(x,y)f_4(x,y) = R(y)$$

$$R(y) = -512(-16+y)(-4+y)^4(2+y)^2 f(y),$$

$$\begin{aligned} f(y) \;=\; & -65536 - 10276864y - 5027392y^2 \\ & +25146656y^3 + 27552272y^4 + 7538528y^5 \\ & -180256y^6 - 27646y^7 + 944y^8 + 21y^9. \end{aligned}$$



Sturm's Theorem proves:

$$R(y) = 0, y > 0 \;\Rightarrow\; y = y_0, 4, 16$$

$$\text{with } \frac{1}{2} < y_0 < \frac{1}{\sqrt{2}}$$

$$\Rightarrow\; \log_2 y = \alpha = \alpha_0, 2, 4$$

$$\text{with } -1 < \alpha_0 < -\frac{1}{2}$$

We can prove $f_4(\alpha_0) \neq 0$ easily.
Thus $f_4(\alpha, 2^{\alpha}) = f_6(\alpha, 2^{\alpha}) = 0 \Rightarrow \alpha = 2, 4$.

$$
\begin{aligned}
L(x,y) = \quad & x && \Big(268435456 + 5704253440y \\
& && -4900519936y^2 - 10788732928y^3 \\
& && +1665391872y^4 - 2044335168y^5 \\
& && -339308448y^6 - 16071552y^7 \\
& && +3559728y^8 - 160420y^9 \\
& && +31166y^{10} + 2482y^{11} \\
& && +31y^{12}\Big) \\
& -4 && \Big(67108864 - 2313158656y \\
& && -803405824y^2 + 6805321728y^3 \\
& && +4795789440y^4 - 1930678368y^5 \\
& && -379246848y^6 - 11684784y^7 \\
& && +1953024y^8 + 113414y^9 \\
& && -4550y^{10} + 2707y^{11} \\
& && +45y^{12}\Big).
\end{aligned}
$$

$$
\begin{aligned}
M(x,y) = \quad & x^3 \Big(12884901888 + 291051143168y \\
& +129899692032y^2 - 865502298112y^3 \\
& -665337083904y^4 + 113529684992y^5 \\
& +12851181696y^6 - 4933789824y^7 \\
& -907026720y^8 + 25947264y^9 \\
& +2714152y^{10} - 299016y^{11} \\
& +79330y^{12} + 3288y^{13} + 31y^{14}\Big) \\
& -2x^2 \Big(50465865728 + 773060558848y \\
& -95409930240y^2 - 2156632178688y^3 \\
& -752601498624y^4 + 570103937280y^5 \\
& -53961508992y^6 - 60918928512y^7 \\
& -6690174624y^8 + 59713360y^9 \\
& +43040792y^{10} - 1087920y^{11} \\
& +657030y^{12} + 33936y^{13} + 369y^{14}\Big) \\
& +8x \Big(14495514624 - 274861129728y \\
& -257627258880y^2 + 562210586624y^3 \\
& +1106047222784y^4 + 651747223040y^5 \\
& -45793043520y^6 - 80250740736y^7 \\
& -9184728480y^8 - 48378720y^9 \\
& +67514796y^{10} - 1554720y^{11} \\
& +605392y^{12} + 54856y^{13} + 715y^{14}\Big) \\
& -32 \Big(1006632960 - 19713228800y \\
& -94432198656y^2 - 4409028608y^3 \\
& +275213442304y^4 + 281906870976y^5 \\
& +33707210784y^6 - 29192050368y^7 \\
& -4054720752y^8 - 83636004y^9 \\
& +22319090y^{10} + 845634y^{11} \\
& +17501y^{12} + 28166y^{13} + 450y^{14}\Big).
\end{aligned}
$$

Conclusion

Theorem: Motion with conditions

$$I = const.,\ L = 0 \text{ and } \mathbf{x}_3(0) = \mathbf{0}$$

is possibe only when

$$\alpha = -2, 2, 4$$

— — — — —

When $\alpha = 2$: Harmonic Oscillator.
$\Rightarrow$ No Figure-Eight

When $\alpha = 4$: $\theta = 0$, i.e., One dimentional motion.
$\Rightarrow$ No Figure-Eight

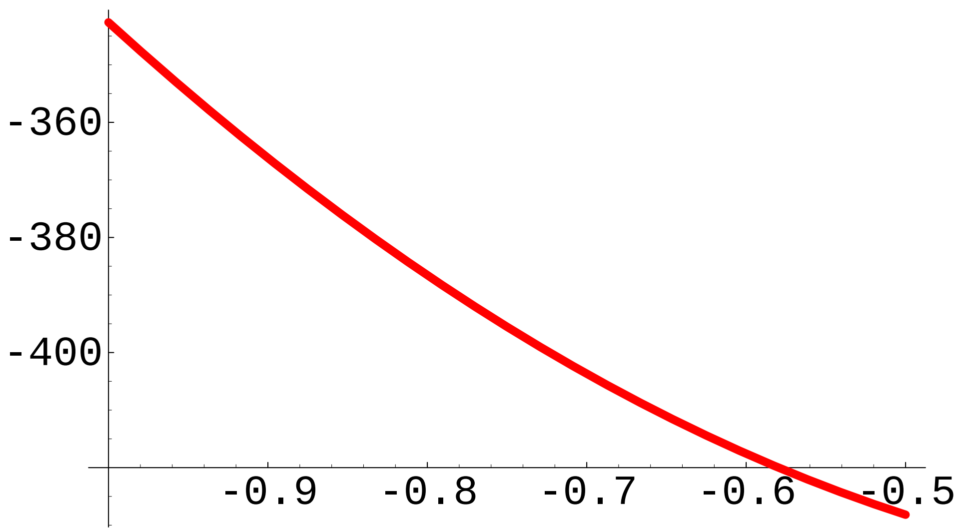
Chenciner's Problem: Figure-Eight with $I = const.$ Only when $\alpha = -2$ is solved.

The followings are added after the conference.

1) We find a minor mistake in our original proof of $f_4(\alpha_0) \neq 0$ in our preprint v.1 and v.2.

Correction is easy.
See new version v.3 (or newer) on arXiv.org

Anyway, we can simply prove
$f_4(\alpha_0) < 0$ for $-1 < \alpha_0 < -1/2$.



2) In our preprint v.2 (or newer), we explain why $\alpha = 2$ and 4 is so special.