Review of Several Motions on the Lemniscate
ver. 1.1

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1 Preparations

1.1 Cartesian Coordinate and Polar Coordinate in 2 Dimensions

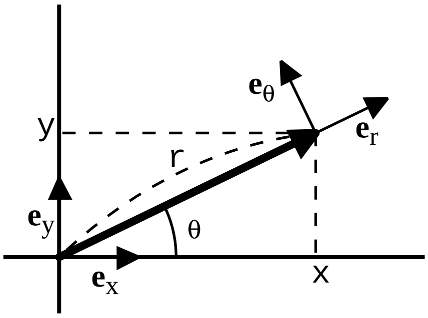

Figure 1: Cartesian Coordinate and Polar Coordinate: Unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ for cartesian coordinate and $\mathbf{e}_r$, $\mathbf{e}_\theta$ for polar coordinate.

Position vector $\mathbf{x}$ is represented by cartesian coordinate

$$\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y == r(\cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y), \tag{1}$$

whereas represented by polar coordinate

$$\mathbf{x} = r\mathbf{e}_r. \tag{2}$$

Unit vectors for polar coordinate are defined as follows,

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y, && (3)\\ \mathbf{e}_\theta &= -\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y. && (4) \end{aligned}$$

Suppose that the position is represented by a parameter. Then orbit $\mathbf{x}$ represents a curve and $\mathbf{x}$, r, θ, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the parameter. Derivative of them by the parameter are

$$\begin{aligned} \dot{\mathbf{e}}_r &= \dot\theta(-\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y) = \dot\theta\mathbf{e}_\theta, && (5)\\ \dot{\mathbf{e}}_\theta &= \dot\theta(-\cos\theta\ \mathbf{e}_x - \sin\theta\ \mathbf{e}_y) = -\dot\theta\mathbf{e}_r, && (6)\\ \dot{\mathbf{x}} &= \dot r\mathbf{e}_r + r\dot\theta\mathbf{e}_\theta, && (7)\\ \ddot{\mathbf{x}} &= (\ddot r - r\dot\theta^2)\mathbf{e}_r + (2\dot r\dot\theta + r\ddot\theta)\mathbf{e}_\theta. && (8) \end{aligned}$$

where dot above symbols represents derivative of them by the parameter.

If the parameter is identified, we will write the derivative explicitly by this parameter. For example, if the parameter is the time, the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are given by

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{x}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta && (9)\\ \mathbf{a} &= \frac{d^2\mathbf{x}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta. && (10) \end{aligned}$$

Another useful example is to take the angle θ as the parameter. In this case,

$$\begin{aligned}\frac{d\mathbf{x}}{d\theta} &= \frac{dr}{d\theta}\mathbf{e}_r + r\mathbf{e}_\theta & (11)\\ \frac{d^2\mathbf{x}}{d\theta^2} &= \left(\frac{d^2r}{d\theta^2} - r\right)\mathbf{e}_r + 2\frac{dr}{d\theta}\mathbf{e}_\theta. & (12)\end{aligned}$$

1.2 Tangent Vector, Principal Normal Vector and the Curvature for 2 Dimensional Curve

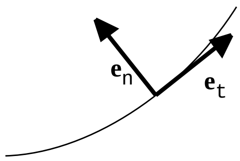

Figure 2: $\mathbf{e}_t$ and $\mathbf{e}_n$.

We can take the length s along the curve from some point as the parameter. Then, $d\mathbf{x}/ds$ is a vector with unit length and tangent to the curve. We write this vector $\mathbf{e}_t$,

$$\mathbf{e}_t = \frac{d\mathbf{x}}{ds}. \tag{13}$$

Second derivative of $\mathbf{x}$ with respective to s is not a unit vector in general. The length of this vector is called the curvature. We write the curvature $1/\rho$,

$$\frac{d\mathbf{e}_t}{ds} = \frac{1}{\rho}\mathbf{e}_n, \tag{14}$$

where the vector $\mathbf{e}_n$ is called principal normal vector. ρ is called radius of curvature.

Since $0 = d(\mathbf{e}_t^2)/ds = 2\mathbf{e}_t \cdot \mathbf{e}_n/\rho$, principal normal vector $\mathbf{e}_n$ is perpendicular to tangent vector $\mathbf{e}_t$,

$$\mathbf{e}_t \cdot \mathbf{e}_n = 0. \tag{15}$$

We take the direction of $\mathbf{e}_n$ such as $\mathbf{e}_t$ and $\mathbf{e}_n$ form a right-hand coordinate. (See fig. 1.2)

Differentiating this equation by s, we get

$$0 = \frac{d\mathbf{e}_t}{ds} \cdot \mathbf{e}_n + \mathbf{e}_t \cdot \frac{d\mathbf{e}_n}{ds} = \frac{1}{\rho} + \mathbf{e}_t \cdot \frac{d\mathbf{e}_n}{ds}. \tag{16}$$

On the other hand, since $0 = d(\mathbf{e}_n^2)/ds = 2\mathbf{e}_n \cdot d\mathbf{e}_n/ds$, $d\mathbf{e}_n/ds$ is proportional to the vector $\mathbf{e}_t$ in 2 dimensions. Therefore, we get

$$\frac{d\mathbf{e}_n}{ds} = -\frac{1}{\rho}\mathbf{e}_t. \tag{17}$$

Equations (14) and (17) are the Serret-Frenet formulas for 2 dimensional curve.

Using a general parameter,

$$\dot{\mathbf{x}} = \dot{s}\frac{d\mathbf{x}}{ds} = \dot{s}\mathbf{e}_t \tag{18}$$

$$\ddot{\mathbf{x}} = \ddot{s}\mathbf{e}_t + \dot{s}\dot{\mathbf{e}}_t = \ddot{s}\mathbf{e}_t + \dot{s}^2\frac{d\mathbf{e}_t}{ds} = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n. \tag{19}$$

Thus, we get

$$\dot{\mathbf{x}} \times \ddot{\mathbf{x}} = \frac{\dot{s}^3}{\rho}\mathbf{e}_t \times \mathbf{e}_n. \tag{20}$$

That is, the curvature is given by a general parametrization as follows,

$$\frac{1}{\rho} = \frac{\dot{\mathbf{x}} \times \ddot{\mathbf{x}}}{\dot{s}^3}. \tag{21}$$

Note that the vector product in 2 dimentions is a (pseudo-)scalar.

For example, if we take time as the parameter, we get

$$\frac{1}{\rho} = \frac{\mathbf{v} \times \mathbf{a}}{v^3}. \tag{22}$$

And, if we take θ as the parameter, we get

$$\frac{1}{\rho} = \left(r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right)\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{-3/2}. \tag{23}$$

1.3 Complex Variable

Sometimes, the complex variable is useful. We use the notation $x^{(+)}$ for $x + i\ y$, and $x^{(-)}$ for $x - i\ y$. Relations among the complex variable , Cartesian and polar coordinate are

$$\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y = r(\cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y) = r\mathbf{e}_r, \tag{24}$$

$$x^{(\pm)} = x \pm i\ y = re^{\pm i\theta}, \tag{25}$$

$$r^2 = \mathbf{x}^2 = x^{(+)}x^{(-)}, \tag{26}$$

$$r = |\mathbf{x}| = \left|x^{(+)}\right| = \left|x^{(-)}\right|. \tag{27}$$

1.4 Jacobian Elliptic Function

Jacobian Elliptic function sn, cn and dn are a generalization of the trigonometric functions. They have two parameters. One of the parameters is called the "modulus" and written as k. In the limit of $k \to 0$, they approach to the trigonometric functions or 1,

$$\mathrm{sn}(u,k) \to \mathrm{sn}(u,0) = \sin(u), \tag{28}$$

$$\mathrm{cn}(u,k) \to \mathrm{cn}(u,0) = \cos(u) \tag{29}$$

$$\mathrm{dn}(u,k) \to \mathrm{dn}(u,0) = 1. \tag{30}$$

They satisfy the following relations,

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1, \tag{31}$$
$$k^2\mathrm{sn}^2(u,k) + \mathrm{dn}^2(u,k) = 1, \tag{32}$$

and differential equations,

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u.k), \tag{33}$$
$$\frac{d}{du}\mathrm{cn}(u,k) = -\mathrm{sn}(u,k)\mathrm{dn}(u,k), \tag{34}$$
$$\frac{d}{du}\mathrm{dn}(u,k) = -k^2\mathrm{sn}(u,k)\mathrm{cn}(u,k). \tag{35}$$

Sometimes, we simply write these functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, $\mathrm{dn}(u)$.

2 The Lemniscate

2.1 Definition of the Lemniscate

The lemniscate in wide meaning is defined as a curve $\mathbf{x}$ on which the product of the distance from given fixed N-points,

$$\prod_{n=1}^{N} |\mathbf{x} - \mathbf{c}_n| = \text{const.}, \tag{36}$$

where c_n's are given fixed points.

Whereas, the lemniscate in narrow meaning is defined as the curve $\mathbf{x}$ on which the product of the distance from two fixed points is constant and pass through the midpoint of the two point. We can take the two points $\pm c/\sqrt{2}\, \mathbf{e}_x$ without loss of generality. Then, the midpoint is the origin. Let us take this form for standard form of the lemniscate,

$$\left|\mathbf{x} - \frac{c}{\sqrt{2}}\mathbf{e}_x\right| \left|\mathbf{x} + \frac{c}{\sqrt{2}}\mathbf{e}_x\right| = \frac{c^2}{2}. \tag{37}$$

To get the relation between the Cartesian coordinate x and y, we may simplify the following equation,

$$\left(\mathbf{x} - \frac{c}{\sqrt{2}}\mathbf{e}_x\right)^2 \left(\mathbf{x} + \frac{c}{\sqrt{2}}\mathbf{e}_x\right)^2 = \frac{c^4}{4}. \tag{38}$$

After a simple calculation, we will get

$$\left(x^2 + y^2\right)^2 = c^2 \left(x^2 - y^2\right). \tag{39}$$

The alternative expression of the definition of the lemniscate is, using the complex variable $x^{(+)}$,

$$\left|\left(x^{(+)}\right)^2 - \frac{c^2}{2}\right| = \frac{c^2}{2}. \tag{40}$$

Then, multiplying the similar expression for $x^{(-)}$, we get

$$\frac{c^4}{4} = \left(\left(x^{(+)}\right)^2 - \frac{c^2}{2}\right)\left(\left(x^{(-)}\right)^2 - \frac{c^2}{2}\right) \tag{41}$$

$$= \left(x^{(+)}x^{(-)}\right)^2 - \frac{c^2}{2}\left(\left(x^{(+)}\right)^2 + \left(x^{(-)}\right)^2\right) + \frac{c^4}{4}. \tag{42}$$

That is,

$$\left(x^{(+)}x^{(-)}\right)^2 = \frac{c^2}{2}\left(\left(x^{(+)}\right)^2 + \left(x^{(-)}\right)^2\right), \tag{43}$$

which is equivalent to the equation (39).

Rewriting the equation (39) or (43) by the polar coordinate, we have another expression for the lemniscate,

$$r^2 = c^2 \cos 2\theta. \tag{44}$$

2.2 Inversion of the Lemniscate and the Rectangular Hyperbola

Consider the following one-to-one map,

$$\mathbf{x} = x\mathbf{e}_x + x\mathbf{e}_y \leftrightarrow \mathbf{X} = X\mathbf{e}_x + Y\mathbf{e}_y = \frac{c^2}{\mathbf{x}^2}\mathbf{x}. \tag{45}$$

Since

$$\mathbf{x} \cdot \mathbf{X} = c^2 \tag{46}$$

and $\mathbf{x}$ and $\mathbf{X}$ have common polar angle, $\mathbf{x}$ and $\mathbf{X}$ are mutual inversion with respective to the circumcircle of the lemniscate. Dividing the both side of the equation (39) by $(x^2 + y^2)^2$, we get

$$\left(\frac{c^2\ x}{x^2 + y^2}\right)^2 - \left(\frac{c^2\ y}{x^2 + y^2}\right)^2 = c^2. \tag{47}$$

That is, the inversion of the lemniscate is the rectangular hyperbola,

$$X^2 - Y^2 = c^2. \tag{48}$$

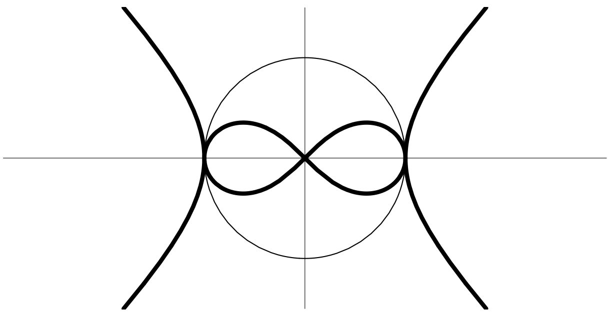

Figure 3: The lemniscate, circle and the rectangular hyperbola.

2.3 Curvature of the Lemniscate

Here, we use the expression of the lemniscate and the curvarure by the polar coordinate,

$$\begin{aligned} r^2 &= c^2 \cos 2\theta, & (49)\\ \frac{1}{\rho} &= \left(r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right)\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{-3/2}. & (50)\end{aligned}$$

Using an identity

$$-r\frac{d^2r}{d\theta^2} = -\frac{1}{2}\frac{d^2}{d\theta^2}r^2 + \left(\frac{dr}{d\theta}\right)^2, \tag{51}$$

and equation for the lemniscate

$$-\frac{1}{2}\frac{d^2}{d\theta^2}r^2 = -\frac{c^2}{2}\frac{d^2}{d\theta^2}\cos 2\theta = 2c^2\cos 2\theta = 2r^2, \tag{52}$$

we get

$$\frac{1}{\rho} = 3\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{-3/2} = 3\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{-1/2}. \tag{53}$$

Using the equation $r^2 = c^2 \cos 2\theta$, we get

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{c^4}{r^2}\sin^2 2\theta = \frac{c^4}{r^2}\left(1 - \frac{r^4}{c^4}\right) = \frac{c^4}{r^2} - r^2. \tag{54}$$

Thus, we finally get the following simple expression of the curvature for the lemniscate,

$$\frac{1}{\rho} = \frac{3r}{c^2}. \tag{55}$$

3 Parameterization of the Lemniscate by Jacobian Elliptic Functions

3.1 Parameterizations of the Lemniscate

Since

$$\left(\frac{1}{\mathrm{sn}(u)}\right)^2 - \left(\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}\right)^2 = \frac{1-\mathrm{cn}^2(u)}{\mathrm{sn}^2(u)} = 1, \tag{56}$$

the vector

$$\mathbf{X}(u) = c\left(\frac{1}{\mathrm{sn}(u)}\mathbf{e}_x + \frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}\mathbf{e}_y\right) \tag{57}$$

is a parametrization of the rectangular hyperbola (48).

Then, the inversion of this vector gives a parametrization of the lemniscate,

$$\mathbf{x}(u) = \frac{c\ \mathrm{sn}(u)}{1+\mathrm{cn}^2(u)}\left(\mathbf{e}_x + \mathrm{cn}(u)\mathbf{e}_y\right). \tag{58}$$

In the complex variable, the lemniscate is parameterized as

$$x^{(\pm)}(u) = \frac{c\ \mathrm{sn}(u)}{1 \mp i\ \mathrm{cn}(u)}. \tag{59}$$

Note that the vector (57) and (57) represents the rectangular hyperbola and the lemniscate respectively, for arbitrary modulus k. For example, taking the limit $k \to 0$,

$$\mathbf{x}(u) \to \frac{c\ \sin(u)}{1+\cos^2(u)}\left(\mathbf{e}_x + \cos(u)\mathbf{e}_y\right) \tag{60}$$

is also a parametrization of the lemniscate.

3.2 Radius, Velocity and Curvature

Let the parameter u be the time t. Then by a simple calculation, we get

$$\begin{aligned}
\mathbf{x}^2(t) &= c^2\frac{\mathrm{sn}^2(t)}{1+\mathrm{cn}^2(t)}, && (61)\\
\mathbf{v}^2(t) &= c^2\frac{\mathrm{dn}^2(t)}{1+\mathrm{cn}^2(t)}, && (62)\\
\mathbf{v}(t)\times\mathbf{a}(t) &= -3c^2\frac{\mathrm{sn}(t)\mathrm{dn}^3(t)}{(1+\mathrm{cn}^2(t))^2}\mathbf{e}_x\times\mathbf{e}_y, && (63)\\
\frac{1}{\rho^2(t)} &= \frac{9}{c^2}\frac{\mathrm{sn}^2(t)}{1+\mathrm{cn}^2(t)}. && (64)
\end{aligned}$$

From this, we get

$$\begin{aligned} \frac{1}{\rho^2(t)} &= \frac{9}{c^4}\mathbf{x}^2(t), && (65) \\ \mathbf{v}^2(t) + \left(k^2 - \frac{1}{2}\right)\mathbf{x}^2(t) &= \frac{c^2}{2}. && (66) \end{aligned}$$

4 Centrifugal Force for the Lemniscate

What centrifugal force drives a motion on the lemniscate?
(Reference: M. R. Spiegel 1967 *Theory and Problems of Theoretical Mechanics* (New York: McGraw-Hill) p 138

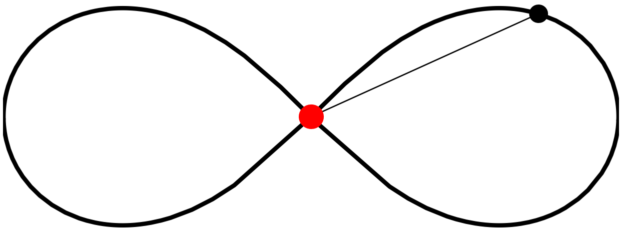

Figure 4: Red circle and black circle represent Sun and a point particle respectively.

4.1 Equation of motion under a Centrifugal Force

Equation of motion under a centrifugal force $f_r \mathbf{e}_r$ is

$$\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta = f_r\mathbf{e}_r. \tag{67}$$

Therefore, equation for the $\mathbf{e}_\theta$ direction is

$$0 = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \tag{68}$$

That is, the angular momentum

$$\ell = r^2\frac{d\theta}{dt} \tag{69}$$

is constant of motion. From the above, we can translate the derivative by time into the derivative by angle θ,

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\ell}{r^2}\frac{d}{d\theta}. \tag{70}$$

Then, the equation of motion for $\mathbf{e}_r$ direction is

$$f_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{r^2}\frac{d}{d\theta}r\right) - \frac{\ell^2}{r^3}. \tag{71}$$

That is, the equation of motion for centrifugal force is

$$\frac{f_r}{\ell^2} + \frac{1}{r^3} = \frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{d}{d\theta}r\right). \tag{72}$$

4.2 Magnitude of the Centrifugal Force for Motion on the Lemniscate

Now, consider motion on the lemniscate $r^2 = c^2 \cos(2\theta)$. Differentiate both side by θ, we get

$$\frac{dr}{d\theta} = -c^2 \frac{\sin 2\theta}{r} = -\frac{\sqrt{c^4 - r^4}}{r}. \tag{73}$$

Using this relation, we can translate the derivative by the angle θ into the the derivative by the radius r,

$$\frac{d}{d\theta} = \frac{dr}{d\theta}\frac{d}{dr} = -\frac{\sqrt{c^4 - r^4}}{r}\frac{d}{dr}. \tag{74}$$

Substituting this expression into the above equation of motion (72), we get

$$\frac{f_r}{\ell^2} + \frac{1}{r^3} = \frac{\sqrt{c^4 - r^4}}{r^3}\frac{d}{dr}\left(\frac{\sqrt{c^4 - r^4}}{r^3}\right) \tag{75}$$

$$= \frac{1}{r^3} - \frac{3c^2}{r^7}. \tag{76}$$

That is, the centrifugal force f_r is

$$f_r = -\frac{3c^2\ell^2}{r^7}. \tag{77}$$

And the potential $U(r)$ that gives this force is

$$U(r) = -\frac{c^2\ell^2}{2r^6}. \tag{78}$$

4.3 Time Dependence

From the polar representation of the lemniscate $r^2 = c^2 \cos 2\theta$ and the conservation of the angular momentum $r^2 d\theta/dt = \ell$,

$$\ell = r^2 \frac{d\theta}{dt} = c^2 \cos 2\theta \, \frac{d\theta}{dt} = \frac{c^2}{2}\frac{d}{dt}(\sin 2\theta). \tag{79}$$

Therefore, we get

$$\sin 2\theta = \frac{2\ell}{c^2} t, \tag{80}$$

or

$$\theta(t) = \frac{1}{2}\sin^{-1}\left(\frac{2\ell}{c^2} t\right), \tag{81}$$

where we take $\theta = 0$ at $t = 0$.

5 Constrained Motion along Arc and Chord of the Lemniscate

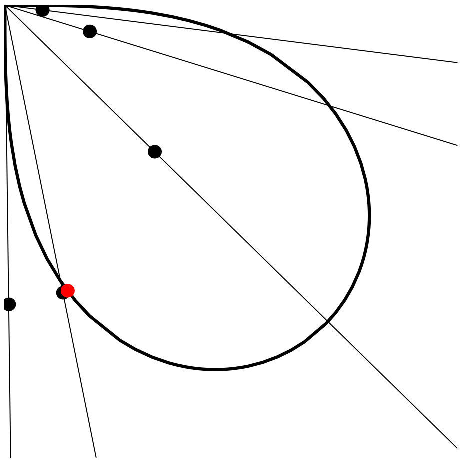

Figure 5: Black circles are motion along chords and red circle is motion along arc.

We don't know who find the following property first. We read this property in Japanese book "Rikigaku Ensyu", edited by Kenichi Goto, Kunio Yamamoto and Ken Kankichi, published by "Kyouritsu Syuppan".

5.1 Quiz and Answer

Quiz: Consider a curve on a vertical plane. A particle is smoothly slide down along this curve under the constant gravitational force. At $t = 0$, this particle starts at the origin O with 0 velocity. Now, for an arbitrarily given point P on this curve, the time to travel the arc OP is the same as the time to travel the chord OP.
What is this curve?

Answer: The lemniscate $r^2 = c^2 \sin 2\theta$, with c arbitrary constant. Where the angle θ is measured from the vertical axis.

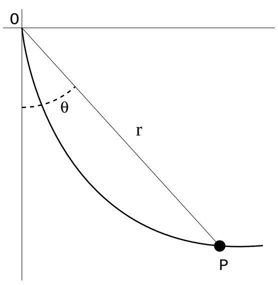

Figure 6: Polar coordinate. The angle θ is measured from the vertical axis.

5.2 Shape of the Curve

Let t be the time at the position P. Considering the motion along the chord OP, we get

$$r = \frac{1}{2} g \cos\theta \; t^2. \tag{82}$$

On the other hand, from the energy conservation of the motion along the arc OP, we get

$$\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 = 2gr\cos\theta. \tag{83}$$

Substituting $\cos\theta$ in equation (82) into equation (83), we get

$$\left(\frac{1}{r}\frac{dr}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = \left(\frac{2}{t}\right)^2. \tag{84}$$

On the other hand, differentiate the logarithm of the the both side of equation (82) by t, we get

$$\frac{1}{r}\frac{dr}{dt} + \tan\theta \; \frac{d\theta}{dt} = \frac{2}{t}. \tag{85}$$

The solution of the above equations (84) and (85) is

$$\frac{1}{r}\frac{dr}{dt} = \cos 2\theta \; \frac{2}{t} \tag{86}$$

$$\frac{d\theta}{dt} = \sin 2\theta \; \frac{2}{t}. \tag{87}$$

Thus we get the following relation between the radius r and the angle θ,

$$2\frac{dr}{r} = \frac{2\cos 2\theta}{\sin 2\theta} d\theta. \tag{88}$$

Integrating this relation, we get

$$r^2 = c^2 \sin 2\theta, \tag{89}$$

with an arbitrary constant c. This is the lemniscate.

5.3 Time dependence

Substituting $r = 1/2 \; g\cos\theta \; t^2$ into $r^2 = c^2 \sin 2\theta$, we get

$$r^2 = 2c^2 \sin\theta\cos\theta = 1/4 \; g^2 \cos^2\theta \; t^4. \tag{90}$$

Therefore,

$$\tan\theta = \frac{g^2}{8c^2} t^4, \tag{91}$$

or

$$\theta(t) = \tan^{-1}\left(\frac{g^2}{8c^2} t^4\right). \tag{92}$$

6 Log Potential from Two Fixed Points

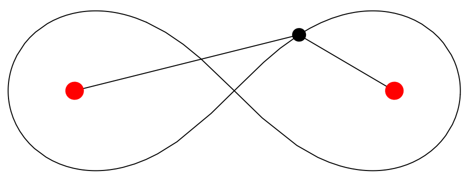

Figure 7: Red circles represents two fixed points and black circle represents a point particle.

6.1 Equation of Motion

Let us consider the motion on the lemniscate

$$\mathbf{x}(t) = c\,\frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}(\mathbf{e}_x + \mathrm{cn}(t)\mathbf{e}_y), \tag{93}$$

with modulus

$$k^2 = \frac{1}{2}. \tag{94}$$

Since

$$\mathbf{v}^2(t) + \left(k^2 - \frac{1}{2}\right)\mathbf{x}^2(t) = \frac{c^2}{2}, \tag{95}$$

for arbitrary modulus k, the speed $v(t)$ is constant for $k^2 = 1/2$,

$$\mathbf{v}^2(t) = \frac{c^2}{2}, \text{ for } k^2 = \frac{1}{2}. \tag{96}$$

On the other hand, potential energy defined by

$$\ln\left|\mathbf{x} - \frac{c}{\sqrt{2}}\mathbf{e}_x\right| + \ln\left|\mathbf{x} + \frac{c}{\sqrt{2}}\mathbf{e}_x\right|. \tag{97}$$

is constant on the orbit (93) by the definition of the lemniscate.

Thus, we expect that a potential energy proportional to the above drives the motion (93). Indeed, the following equation of motion is satisfied,

$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{3c^2}{8}\left(\frac{c/\sqrt{2}\;\mathbf{e}_x - \mathbf{x}(t)}{\left(c/\sqrt{2}\;\mathbf{e}_x - \mathbf{x}(t)\right)^2} + \frac{-c/\sqrt{2}\;\mathbf{e}_x - \mathbf{x}(t)}{\left(-c/\sqrt{2}\;\mathbf{e}_x - \mathbf{x}(t)\right)^2}\right). \tag{98}$$

A proof is given soon later.

Potential energy U that gives this equation of motion is

$$U(\mathbf{x}) = \frac{3c^2}{8}\left(\ln\left|\mathbf{x} - \frac{c}{\sqrt{2}}\mathbf{e}_x\right| + \ln\left|\mathbf{x} + \frac{c}{\sqrt{2}}\mathbf{e}_x\right|\right). \tag{99}$$

That is, attractive logarithmic potential from two fixed point drives the motion on the lemniscate.

Note that, since

$$\ddot{\mathbf{x}} = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n \tag{100}$$

for arbitrary parameter and

$$\frac{d^2 s}{dt^2} = 0, \tag{101}$$

$$\left(\frac{ds}{dt}\right)^2 = \mathbf{v}^2 = \frac{c^2}{2}, \tag{102}$$

$$\frac{1}{\rho} = \frac{3r}{c^2}, \tag{103}$$

the acceleration or the force of this motion is simply given by

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \frac{3}{2} r\ \mathbf{e}_n. \tag{104}$$

That is, the magnitude of the the acceleration or the force of this motion is proportional to the distance from the origin r, and the direction is proportional to the principal normal vector $\mathbf{e}_n$.

We find this motion on the lemniscate by the above considerations. But, this simple and beautiful motion must already be found by some other person in old days. After searching several books, we find a description of this motion in Japanese book "Rikigaku Ensyu", edited by Kenichi Goto, Kunio Yamamoto and Ken Kankichi, published by "Kyouritsu Syuppan". This book gives another proof of this motion. We don't know who find this property first.

6.2 Proof

It is convenient to use complex variable $x^{(\pm)}(t) = \mathbf{x}(t) \cdot (\mathbf{e}_x \pm i\ \mathbf{e}_y)$,

$$x^{(\pm)}(t) = c\ \frac{\mathrm{sn}(t)}{1 \mp i\ \mathrm{cn}(t)}. \tag{105}$$

Using the relation of $\mathrm{sn}^2(t) + \mathrm{cn}^2(t) = 1$, $k^2\mathrm{sn}^2(t) + \mathrm{dn}^2(t) = 1$ and $k^2 = 1/2$, we get

$$\frac{d^2}{dt^2} x^{(+)}(t) = -c\ \frac{3(1 + i\ \mathrm{cn}(t))\mathrm{sn}(t)}{2(1 - i\ \mathrm{cn}^2(t))^2} \tag{106}$$

and

$$\frac{1}{c/\sqrt{2} - x^{(-)}(t)} + \frac{1}{-c/\sqrt{2} - x^{(-)}(t)} = -\frac{4(1 + i\ \mathrm{cn}(t))\mathrm{sn}(t)}{c\ (1 - i\ \mathrm{cn}^2(t))^2}. \tag{107}$$

Thus we get the equation of motion

$$\frac{d^2}{dt^2} x^{(+)}(t) = \frac{3c^2}{8} \left(\frac{1}{c/\sqrt{2} - x^{(-)}(t)} + \frac{1}{-c/\sqrt{2} - x^{(-)}(t)} \right) \tag{108}$$

which is equivalent to the equation in (98).

7 Three Body Choreography on the lemniscate

Hiroshi Fukuda, Hiroshi Ozaki and I showed that three body choreography on the lemniscate,

$$\mathbf{x}(t) = \frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}(\mathbf{e}_x + \mathrm{cn}(t)\mathbf{e}_y), \tag{109}$$

where sn and cn are the Jacobian elliptic function with modulus

$$k^2 = \frac{2+\sqrt{3}}{4}, \tag{110}$$

satisfies an equation of motion.

See our preprint "Choreographic Three Bodies on the Lemniscate" for detail.