Synchronised Similar Triangles for Three Body Orbit with Zero Angular Momentum

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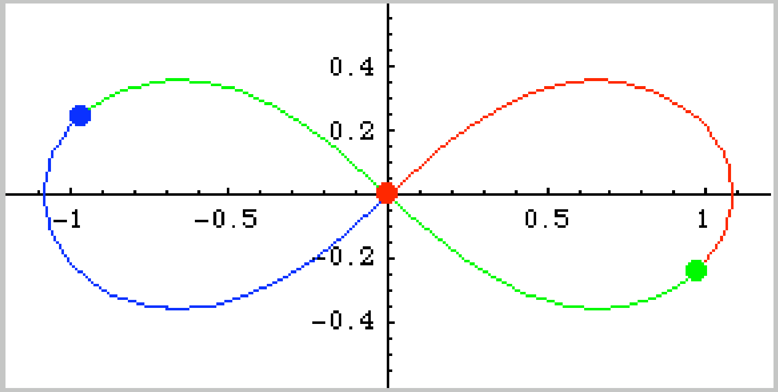
Contents
Geometry of three-body, L=0 orbit

- Motivation and history.
  - The figure-eight 3-body solution
  - Geometrical property of L=0 orbit
- Geometrical property of L=0 and I=const. orbit
- Geometrical property of L=0 orbit

*http://arxiv.org/abs/**math-ph/0404056***
*http://**www.clas.kitasato-u.ac.jp/~fujiwara/**nBody/nbody.html*

Three-body figure-eight choreography

- C. Moore (1993): found numerically
- A. Chenciner and
  R. Montgomery (2000):
  proved the existence



- C. Simó (2000):
  found a lot of N-body choreographies numerically
- Barutello, Ferrario, Terracini,   Chen,   Shibayama, ...

Three-body figure-eight choreography

$$i = 1, 2, 3, \; m_i = 1$$

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{|q_j - q_i|^3},$$

$$\begin{cases} q_1(t) & = q(t), \\ q_2(t) & = q(t + T/3), \\ q_3(t) & = q(t + 2T/3), \end{cases}$$

$$\sum_i q_i = 0, \; L = \sum_i q_i \wedge \dot{q}_i = 0.$$

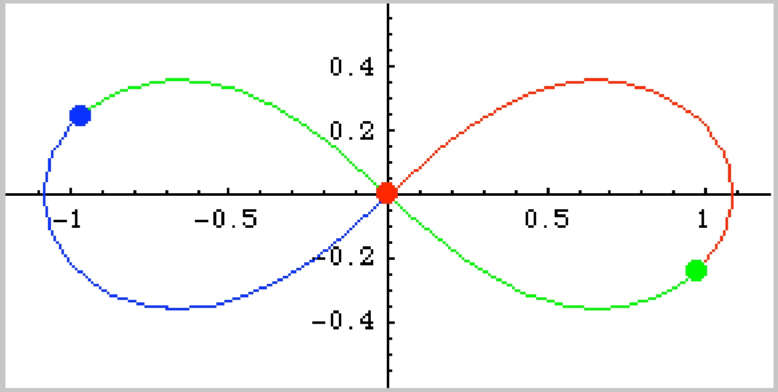


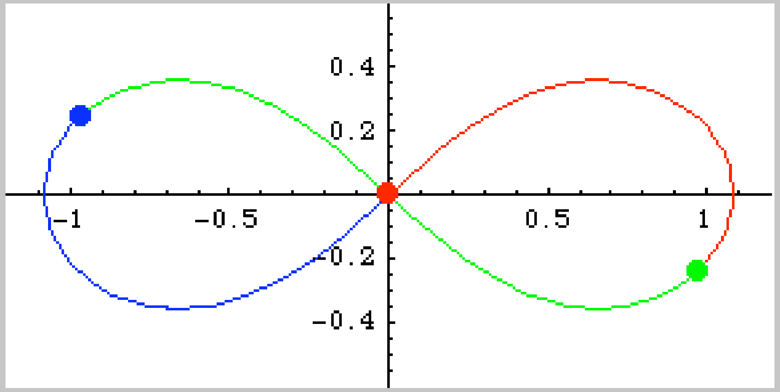
Figure-eight has zero angular momentum

Why $L = 0$?

Total angular momentum is conserved.
Therefore,

$$\sum_i q_i \wedge \dot{q}_i = \sum_i < q_i \wedge \dot{q}_i >= 0.$$

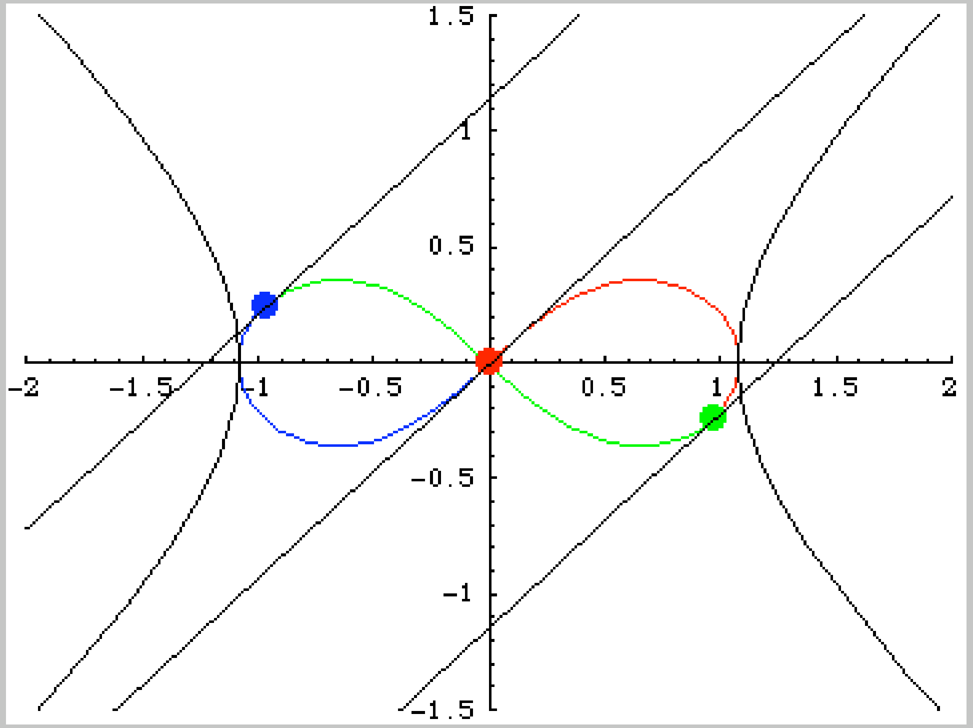
$< \quad >$: *time average*



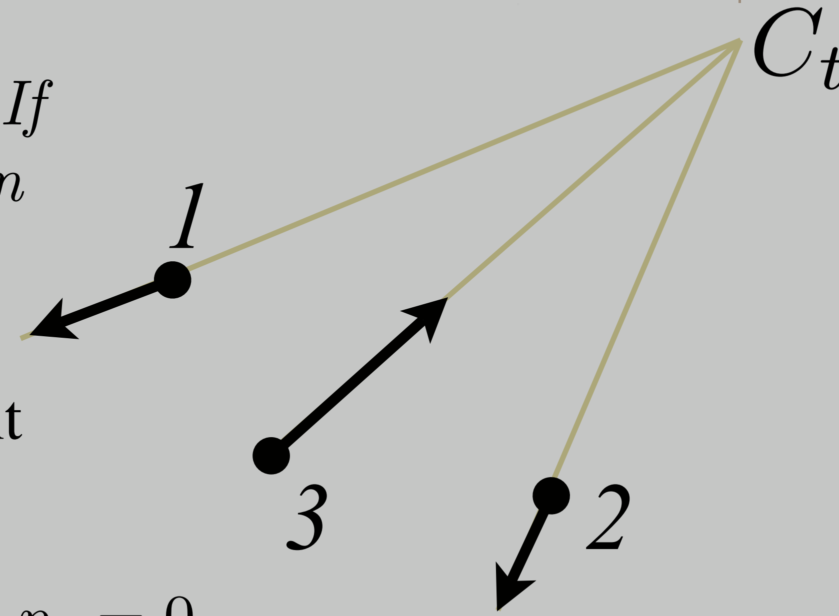
Then, what does $L = 0$ mean?

Three tangents theorem (FFO 2003)
L=0

Theorem (Three Tangents). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \wedge p_i = 0$*, then three tangents meet at a point.*



Three tangents theorem



Theorem (Three Tangents). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \wedge p_i = 0$, *then three tangents meet at a point.*

Proof. Let C_t be the crossing point of two tangent lines p_1 and p_2.

$\therefore (q_1 - C_t) \wedge p_1 = 0, (q_2 - C_t) \wedge p_2 = 0$

On the other hand, $\sum_i p_i = 0, \sum_i q_i \wedge p_i = 0 \Rightarrow \sum_i (q_i - C_t) \wedge p_i = 0.$

$\therefore (q_3 - C_t) \wedge p_3 = 0.$ $\square$

C_t: the "Center of Tangents"

Centre of force for three body orbit dL/dt=0

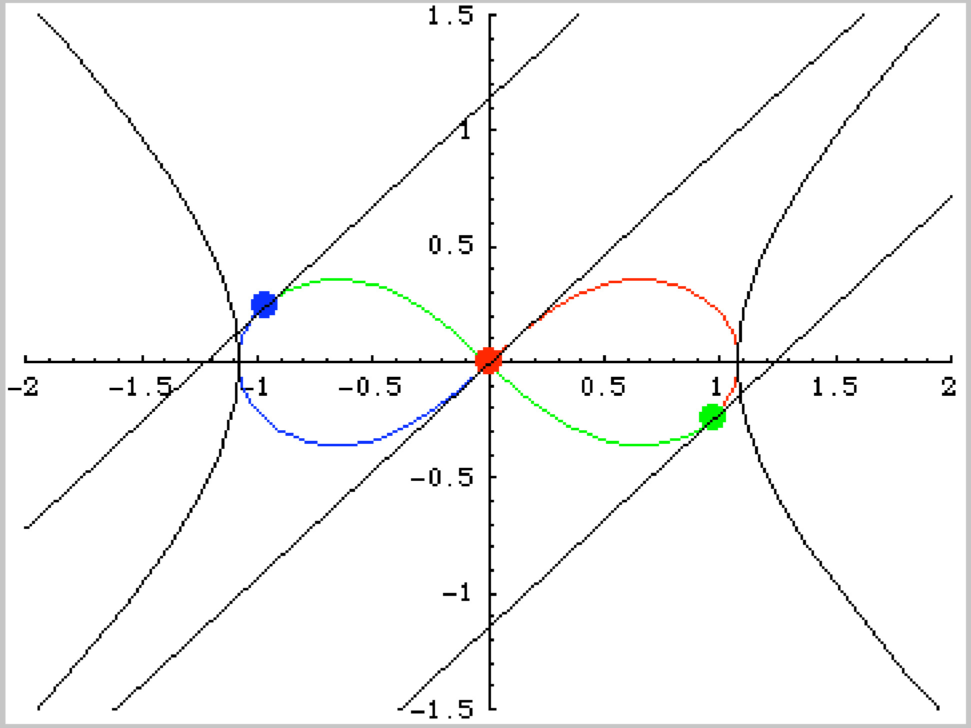
Since

$$\sum_i f_i = 0, \quad \frac{dL}{dt} = \sum_i q_i \wedge f_i = 0,$$

force vector from each bodies meet at a point C_f: the Centre of force. (Schiaparelli, Wintner)

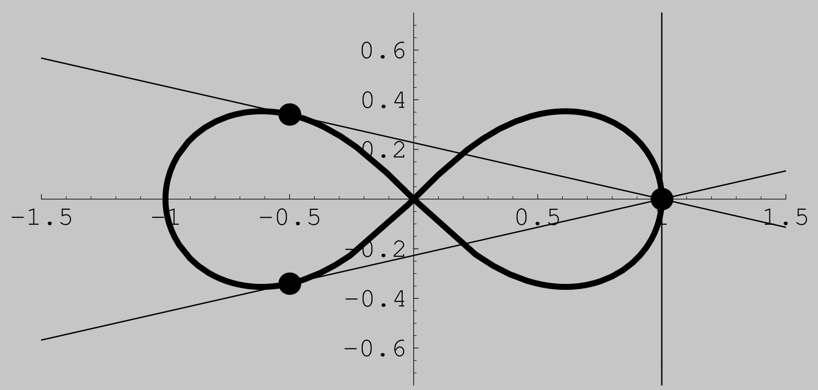
Three tangents theorem
L=0

- Shape of the orbit of Figure Eight x(t) and the orbit C(t) are still unknown.
- Three Tangents Theorem gives a criterion for the orbit.
- For example ...

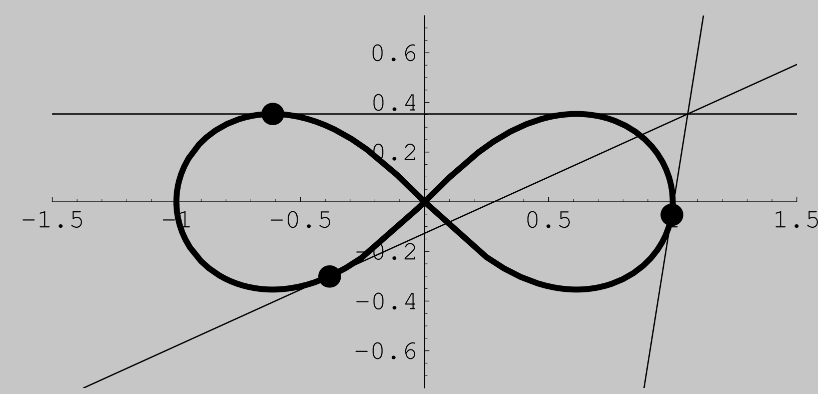


The simplest curve:
4-th order polynomial

$$x^4 + \alpha x^2 y^2 + \beta y^4 = x^2 - y^2$$



$$\alpha = 2$$



$$\beta = 1$$

(numerivally)

Candidate:
Lemniscate
and its scale transform

$$(x^2 + y^2)^2 = x^2 - y^2$$
$$x \to \mu x, y \to \nu y$$

Three-body choreography on the lemniscate (FFO 2003)

Choreograpgy on the Lemniscate

$$q(t) = \left(\frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}, \frac{\mathrm{sn}(t)\mathrm{cn}(t)}{1+\mathrm{cn}^2(t)} \right) \text{ with } k^2 = \frac{2+\sqrt{3}}{4},$$

$$\begin{cases} q_1(t) = q(t), \\ q_2(t) = q(t+T/3), \\ q_3(t) = q(t+2T/3), \end{cases}$$

satisfies the equation of motion $\ddot{q}_i = -\frac{\partial}{\partial q_i} U$ with

$$U = \sum_{i<j} \left(\frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2 \right).$$

Figure-eight solution under homogeneous potential

$$V_\alpha = \begin{cases} \alpha^{-1}\, r^\alpha & \text{for } \alpha \neq 0 \\ \log r & \text{for } \alpha = 0 \end{cases}$$

Numerical evidence

Moore: Exist for $\alpha < 2$

CGMS: Exist for $\alpha < 0$ and Stable $\alpha = -1 \pm \epsilon$

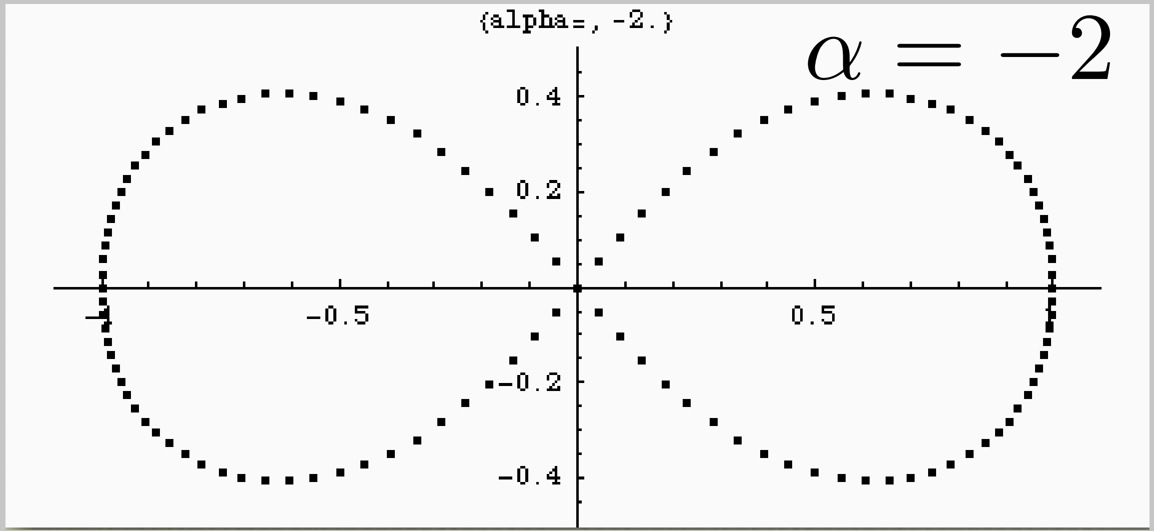


Figure-eight for $\alpha = -2$ has $I = \sum_i q_i^2 = \text{const.}$

Evolution of moment of inertia

$$\sum_i m_i q_i = 0, \quad M = \sum_i m_i, \quad I = \sum_i m_i q_i^2,$$

$$K = \sum_i \dot{q}_i^2, \quad V_\alpha = \frac{1}{\alpha} \sum_{i<j} m_i m_j r_{ij}^\alpha, \quad H = \frac{1}{2} K + V_\alpha.$$

$$\Rightarrow \frac{d^2 I}{dt^2} = 2K - 2\alpha V_\alpha = 4E - 2(2+\alpha) V_\alpha.$$

Lagrange-Jacobi identity

For $\alpha = -2 : \quad \frac{d^2 I}{dt^2} = 4E \;\Rightarrow\; I = 2Et^2 + c_1 t + c_2.$

$\therefore$ If $I \nrightarrow 0$ and $I \nrightarrow \infty$ then $E = 0,\; c_1 = 0,\; I = \text{const.}$

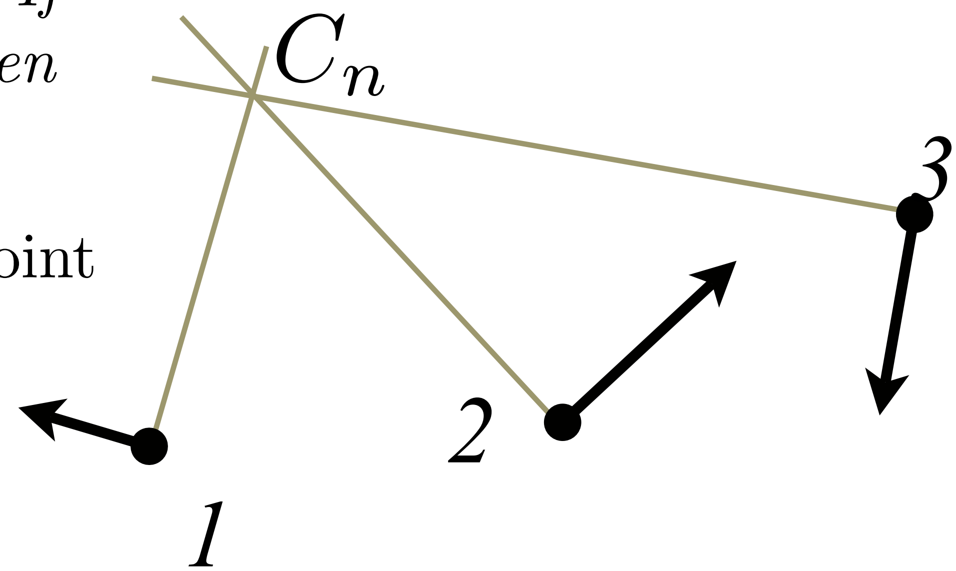
Figure-eight under 1/r^2 has I=constant.
What does this mean?

Three normals theorem for dI/dt=0 orbit

Theorem (Three Normals). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then three normals meet at a point.*

Proof. Let C_n be the crossing point of two normals n_1 and n_2.

Then, $\sum_i (q_i - C_n) \cdot p_i = 0$,
$(q_1 - C_n) \cdot p_1 = 0$ and
$(q_2 - C_n) \cdot p_2 = 0$.
$\therefore\ (q_3 - C_n) \cdot p_3 = 0$. $\square$

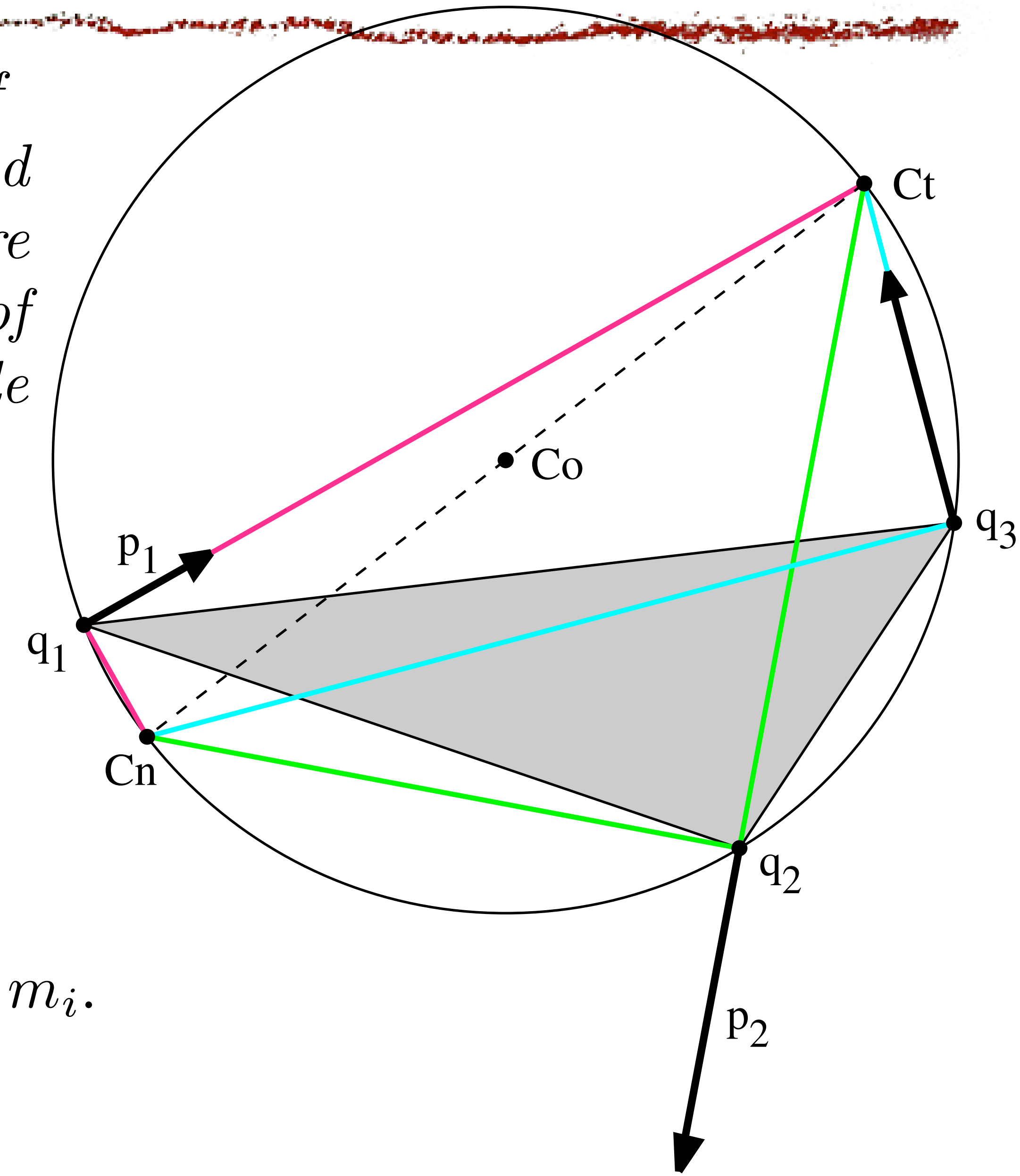


This theorem holds for general masses m_i.

C_n: the "Center of Normals"

Circumcircle theorem for L=0, dI/dt=0 orbit

Theorem (CircumCircle). *If* $\sum_i p_i = 0$, $\sum_i q_i \wedge p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then* C_t *and* C_n *are the end points of a diameter of the circumcircle for the triangle* $q_1 q_2 q_3$.

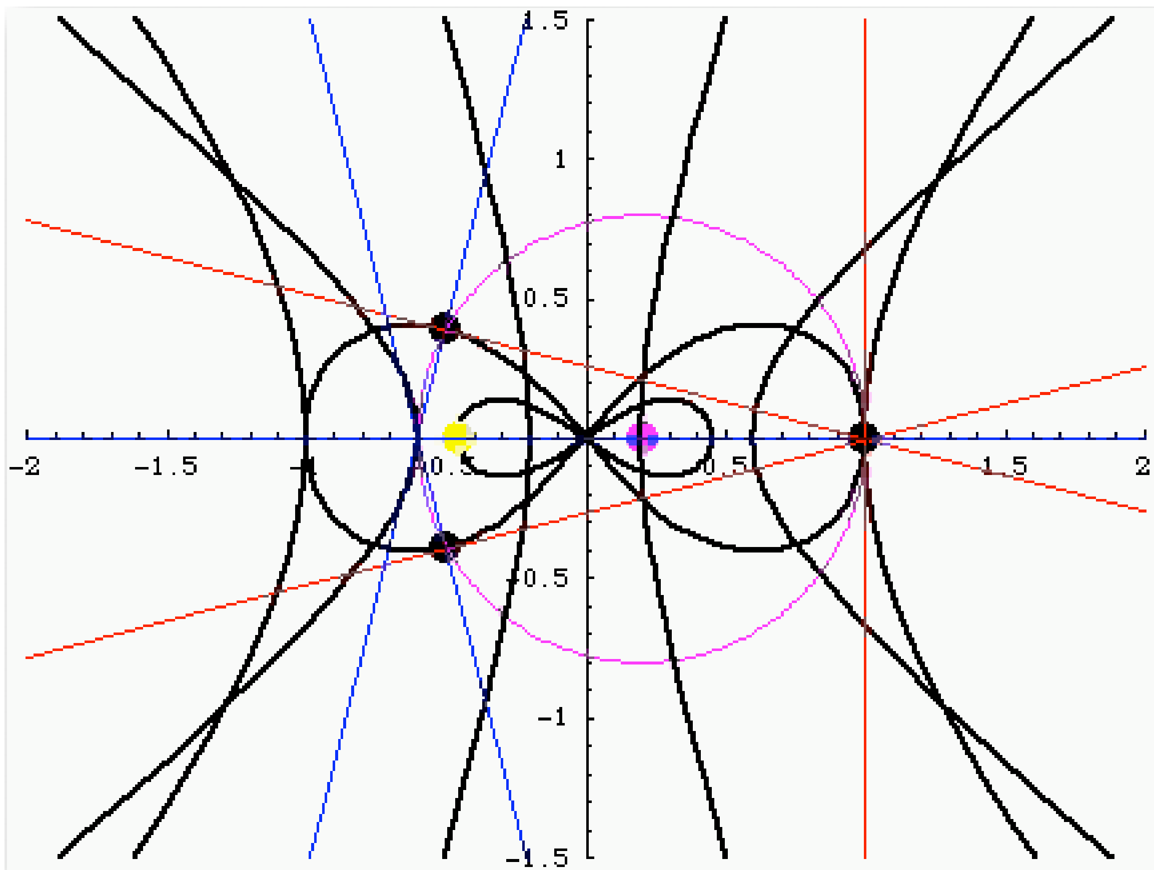


Proof. Angles $C_t q_i C_n$ are 90 degrees for $i = 1, 2, 3$. $\square$

This theorem holds for any masses m_i.

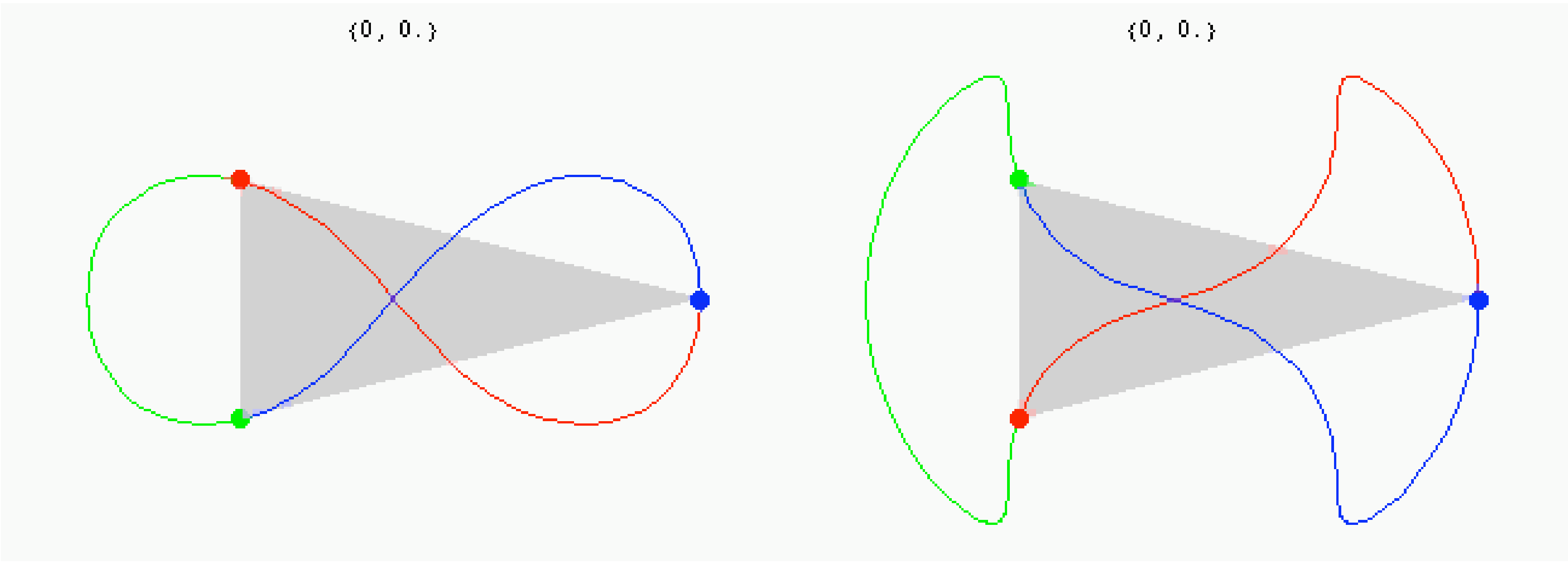
Centres for figure-eight solution under 1/r^2 potential

Figure-eight $\Rightarrow L = 0.$ $\quad V = \dfrac{1}{r^2} \Rightarrow I = \text{const.}$



What does this mean ?...

Synchronised similar triangles for figure-eight under 1/r^2



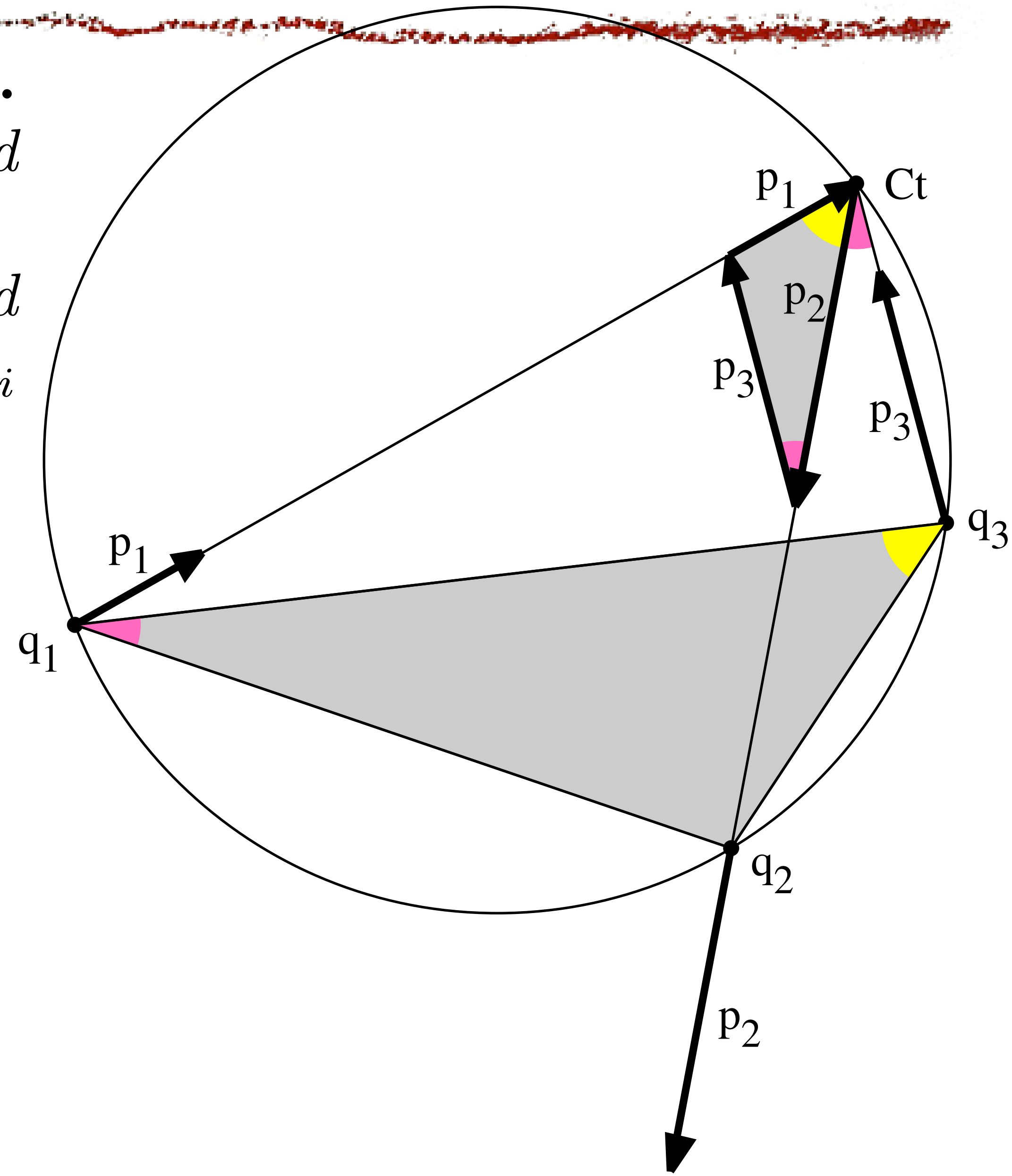
$$q_i' = \frac{q_i}{\sqrt{I}} \qquad m_i = 1 \qquad p_i' = \frac{p_j - p_k}{\sqrt{3K}}$$

$$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$$

Two triangles are inversely congruent.
Because ...

Similar triangles for L=0, dI/dt=0 orbit.

Theorem (Similar Triangles).
If $\sum_i p_i = 0$, $\sum_i q_i \wedge p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then triangle whose vetices are* q_i *and triangle whose perimeters are* p_i *are similar with reverse orientation.*



Proof. Look at the angles yellow colored and red colored.
It is obvious. □

Remark: *This theorem holds for any masses* m_i

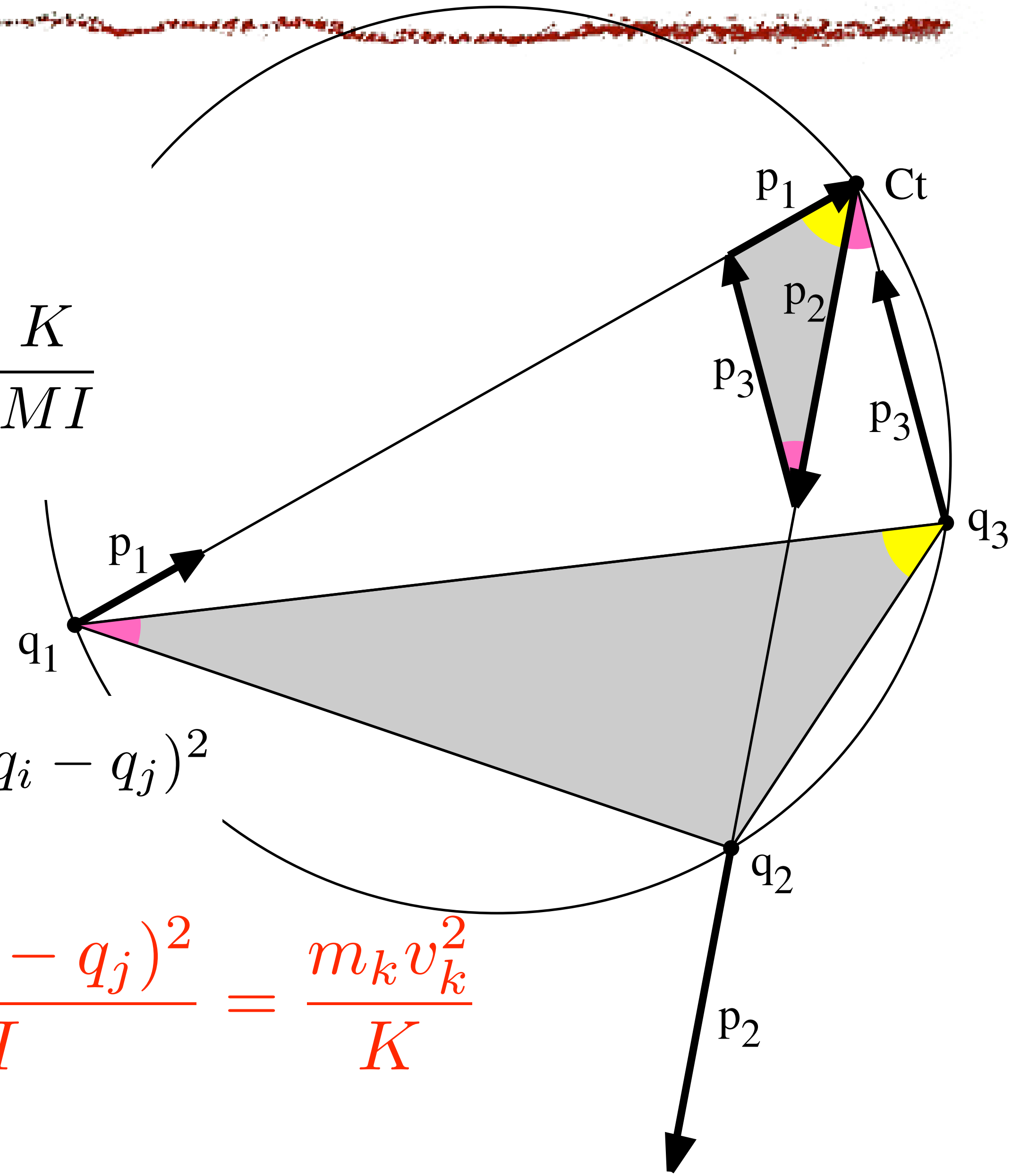
Ratio of magnification
(L=0, dI/dt=0)

$$\kappa(t) = \frac{|p_k|}{|q_i - q_j|} = \sqrt{\frac{m_1 m_2 m_3 K}{M I}}$$

$$\because \frac{\kappa(t)^2}{m_1 m_2 m_3} = \frac{p_k^2/m_k}{m_i m_j (q_i - q_j)^2} = \frac{K}{M I}$$

where $K = \sum_i \frac{p_i^2}{m_i}$, $M = \sum_i m_i$,

$I = \sum_i m_i q_i^2 = M^{-1} \sum_{i<j} m_i m_j (q_i - q_j)^2$



$$\therefore \frac{m_i m_j (q_i - q_j)^2}{M I} = \frac{m_k v_k^2}{K}$$

Oriented area
(L=0, dI/dt=0)

$$
\begin{aligned}
p_1 \wedge p_2 &= -k^2(q_2 - q_1) \wedge (q_3 - q_1) \\
&= -\frac{K}{MI} m_1 m_2 m_3 (q_1 \wedge q_2 + q_2 \wedge q_3 + q_3 \wedge q_1) \\
&= -\frac{K}{I} m_1 m_2 q_1 \wedge q_2.
\end{aligned}
$$

$\because \sum_i m_i q_i = 0 \Rightarrow m_1 m_2 q_1 \wedge q_2 = m_2 m_3 q_2 \wedge q_3 = m_3 m_1 q_3 \wedge q_1.$

$$
\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = 0.
$$

Energy balance for L=0, dI/dt=0 orbit under 1/r^2

$$\frac{d^2 I}{dt^2} = 0 \Rightarrow K = \sum_{i<j} \frac{m_i m_j}{r_{ij}^2}$$

$$L = 0, \quad \frac{dI}{dt} = 0 \Rightarrow \frac{1}{r_{ij}^2} = \frac{m_1 m_2 m_3 K}{MI} \, \frac{1}{p_k^2}.$$

$$\therefore K = \frac{m_1 m_2 m_3 K}{MI} \left(\frac{m_1 m_2}{p_3^2} + \frac{m_2 m_3}{p_1^2} + \frac{m_3 m_1}{p_2^2} \right)$$

$$\therefore \frac{m_1 m_2}{p_3^2} + \frac{m_2 m_3}{p_1^2} + \frac{m_3 m_1}{p_2^2} = \frac{MI}{m_1 m_2 m_3} = \text{const.}$$

Geometrical properties of L=0 orbit

- **In the followings, we consider L=0 orbit, not assuming dI/dt=0.**
- Even in this case, we can find the synchronised similar triangles.

Synchronised similar triangles for L=0 orbit

For $L = \sum_i m_i q_i \wedge v_i = 0$ but $\frac{dI}{dt} = \sum_i m_i q_i \cdot v_i \neq 0$ orbit, consider

$$\xi_i = \frac{q_i}{\sqrt{I}}, \ \eta_i = \frac{d\xi_i}{dt} = \frac{v_i}{\sqrt{I}} - \frac{1}{2I}\frac{dI}{dt}\frac{q_i}{\sqrt{I}}, \ \mu_i = m_i.$$

Then, we have

$$\sum_i \mu_i \xi_i = 0, \sum_i \mu_i \eta_i = 0, \sum_i \mu_i \xi_i \wedge \eta_i = 0, \sum_i \mu_i \xi_i \cdot \eta_i = 0.$$

$\therefore$ Triangle whose vertexes are $\xi_i = \dfrac{q_i}{\sqrt{I}}$ and

triangle whose perimeters are $\mu_i \eta_i = \mu_i \dfrac{d\xi_i}{dt}$ are always inversely similar.

(Synchronised Similar Triangles)

Purely algebraic derivation of synchronised similar triangles

Let $i = 1, 2, 3$, $\xi_i, \eta_i \in \mathbb{R}^2$, $\mu_i > 0$ such that

$$\sum_i \mu_i \xi_i = 0, \sum_i \mu_i \eta_i = 0, \sum_i \mu_i \xi_i \wedge \eta_i = 0, \sum_i \mu_i \xi_i \cdot \eta_i = 0.$$

Let $M = \sum_i \mu_i$, $I(\xi) = \sum_i \mu_i \xi_i^2 = M^{-1} \sum_{i<j} \mu_i \mu_j (\xi_i - \xi_j)^2$. Then

$$\frac{\mu_k \xi_k^2}{I(\xi)} = \frac{\mu_i \mu_j (\eta_i - \eta_j)^2}{M I(\eta)}, \qquad \frac{\mu_k \eta_k^2}{I(\eta)} = \frac{\mu_i \mu_j (\xi_i - \xi_j)^2}{M I(\xi)},$$

$$\frac{\mu_k \xi_k^2}{I(\xi)} + \frac{\mu_k \eta_k^2}{I(\eta)} = \frac{\mu_i \mu_j (\xi_i - \xi_j)^2}{M I(\xi)} + \frac{\mu_i \mu_j (\eta_i - \eta_j)^2}{M I(\eta)} = \frac{\mu_i + \mu_j}{M}$$

and

$$\frac{\xi_i \wedge \xi_j}{I(\xi)} + \frac{\eta_i \wedge \eta_j}{I(\eta)} = 0.$$

Synchronised similar triangles for L=0 orbit

Therefore, we get

$$\frac{\xi_i \wedge \xi_j}{I(\xi)} + \frac{\eta_i \wedge \eta_j}{I(\eta)} = 0$$

where

$$\xi_i = \frac{q_i}{\sqrt{I}},\ \eta_i = \frac{d\xi_i}{dt} = \frac{v_i}{\sqrt{I}} - \frac{1}{2I}\frac{dI}{dt}\frac{q_i}{\sqrt{I}}.$$

That is

$$\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = \frac{1}{2IK}\frac{dI}{dt}\frac{d}{dt}(q_i \wedge q_j)$$

where

$$I = \sum_i m_i q_i^2,\ K = \sum_i m_i v_i^2.$$

Evolution of oriented area for L=0 orbit

Let

$$\Delta = \frac{1}{2}(q_2 - q_1) \wedge (q_3 - q_1) = \sum_{i<j} (q_i \wedge q_j).$$

Then

$$\frac{d^2}{dt^2}(q_i \wedge q_j) = \frac{d^2 q_i}{dt^2} \wedge q_j + q_i \wedge \frac{d^2 q_j}{dt^2} + 2 v_i \wedge v_j.$$

$$v_i \wedge v_j = -\frac{K}{I}(q_i \wedge q_j) + \frac{1}{2I}\frac{dI}{dt}\frac{d}{dt}(q_i \wedge q_j)$$

$$\therefore I \frac{d}{dt}\left(\frac{1}{I}\frac{d\Delta}{dt}\right) = -\left(\frac{2K}{I} + \sum_{i<j}(m_i + m_j) r_{ij}^{\alpha-2}\right)\Delta.$$

Infinitely many syzygies or collisions (Montgomery 2002)

- Montgomery formulated and proved:
  Any bounded three-body orbit with L=0 has infinitely many collinear configurations (syzygies or eclipses) or collisions.

- We give a simple proof.

A simple proof of infinitely many syzygies or collisions

Proof. Let

$$S(t) = \frac{\Delta(t)}{\sqrt{I(t)}}.$$

Then

$$\frac{d^2 S}{dt^2} = -\left\{ \sum_{i<j} (m_i + m_j) r_{ij}^{\alpha-2} + \frac{2K}{I} + \frac{1}{2I}\frac{d^2 I}{dt^2} - \frac{3}{4I^2}\left(\frac{dI}{dt}\right)^2 \right\} S$$
$$= -\omega^2 S,$$

$$\omega^2 \geq \omega_0^2 > 0 \text{ where } \omega_0^2 = M \left(\frac{m_{min}^2}{M I_{max}}\right)^{(2-\alpha)/2} \quad \text{for } \alpha \leq 2.$$

□

Conclusion 1: Synchronised similar triangles for L=0, dI/dt=0 orbit.

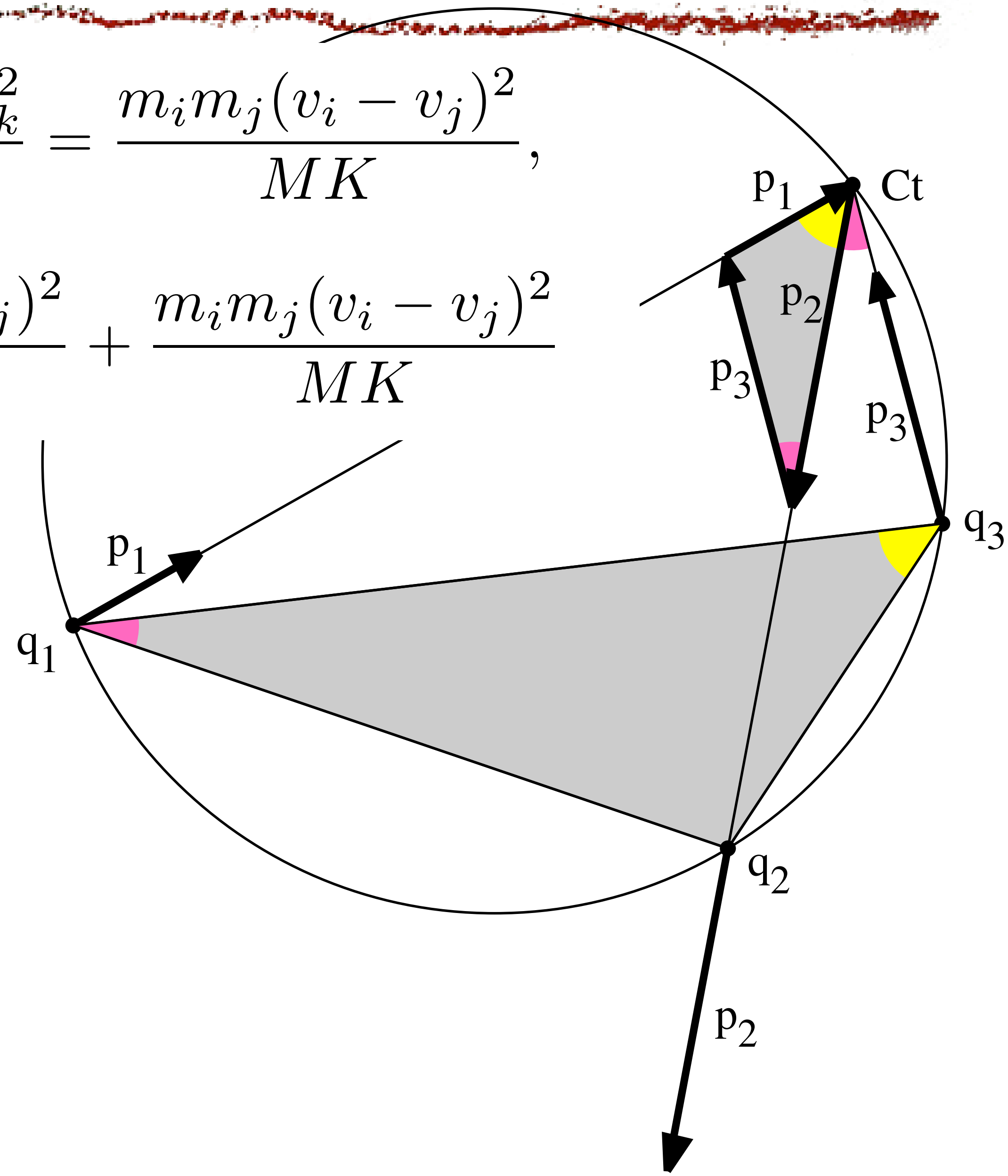
$$\frac{m_i m_j (q_i - q_j)^2}{MI} = \frac{m_k v_k^2}{K}, \quad \frac{m_k q_k^2}{I} = \frac{m_i m_j (v_i - v_j)^2}{MK},$$

$$\frac{m_k q_k^2}{I} + \frac{m_k v_k^2}{K} = \frac{m_i m_j (q_i - q_j)^2}{MI} + \frac{m_i m_j (v_i - v_j)^2}{MK}$$

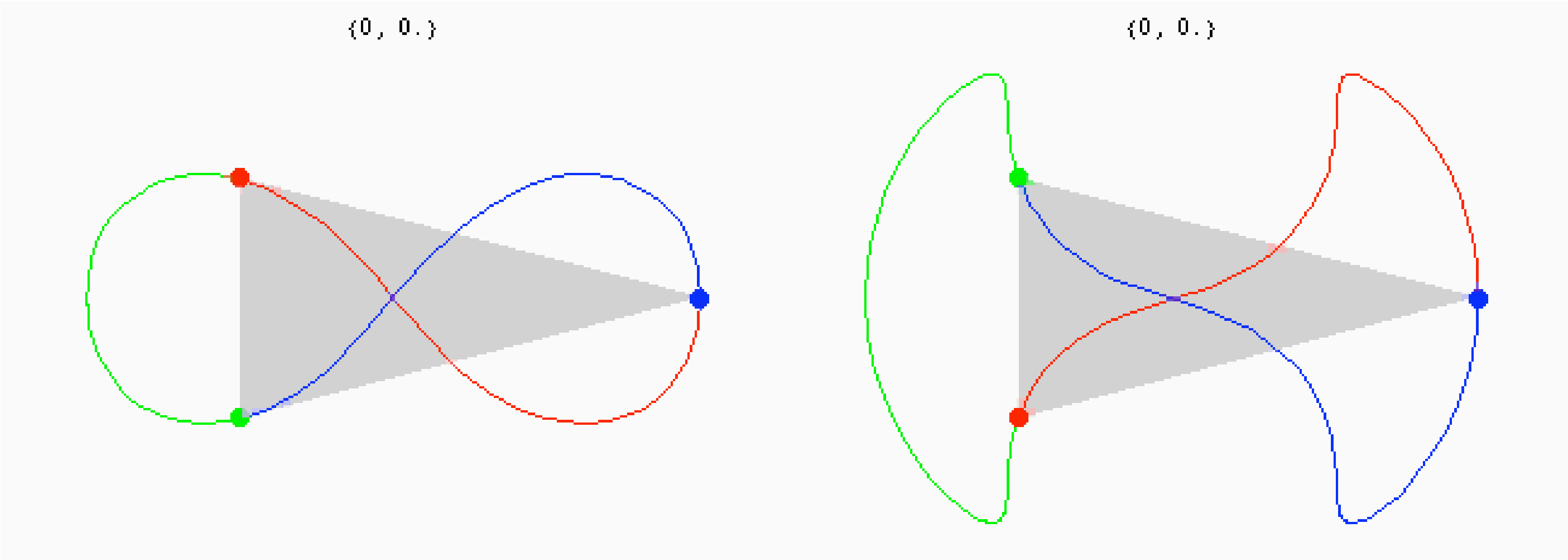
$$= \frac{m_i + m_j}{M},$$

$$\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = 0,$$

$$\sum_{ijk} m_i m_j |p_k|^\alpha = \text{const.}$$



Conclusion 2: Synchronised similar triangles for figure-eight under 1/r^2



$$q_i' = \frac{q_i}{\sqrt{I}} \qquad m_i = 1 \qquad p_i' = \frac{p_j - p_k}{\sqrt{3K}}$$

Two triangles are inversely congruent.

$$\sum_i \frac{1}{p_i^2} = 3I$$

Conclusion 3: Synchronised similar triangles for L=0 orbit.

The variables form the synchronised similar triangles

$$\xi_i = \frac{q_i}{\sqrt{I}},\ \eta_i = \frac{d\xi_i}{dt} = \frac{v_i}{\sqrt{I}} - \frac{1}{2I}\frac{dI}{dt}\frac{q_i}{\sqrt{I}}.$$

Then, we get

$$\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = \frac{1}{2IK}\frac{dI}{dt}\frac{d}{dt}(q_i \wedge q_j),$$

$$I\frac{d}{dt}\left(\frac{1}{I}\frac{d\Delta}{dt}\right) = -\left(\frac{2K}{I} + \sum_{i<j}(m_i + m_j)r_{ij}^{\alpha-2}\right)\Delta.$$

We gave a short proof that $\Delta(t)$ has infinitely many zeros if $\alpha \leq 2$.

I have a dream.
One day,
someone will e-mail me

and say

"I have *solved* the figure-eight !"

Thank you.