Three-Body Figure-Eight Choreography

Shape and Time Evolution

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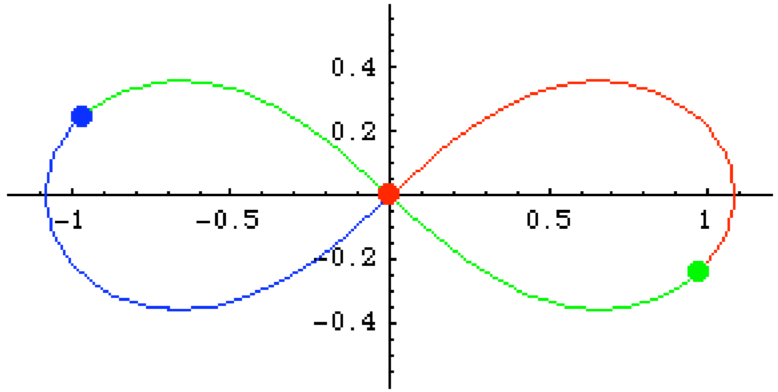
Contents

- Three tangents theorem *(FFO1)*
- Three-body choreography on the lemniscate *(FFO1)*
- Inconstancy of the moment of inertia *(FFO2)*
- Convexity of each lobe *(FM)*
- Open questions *(FFHO working …)*

Three-Body Figure-Eight Choreography

- C. Moore (1993): finds numerically
- A. Chenciner and R. Montgomery (2000): prove the existence
- C. Simó (2000): finds lots of N-body choreography numerically



Three-Body Figure-Eight Choreography

$$i = 1, 2, 3,\ m_i = 1$$

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{|q_j - q_i|^3},$$

$$\begin{cases} q_1(t) = q(t), \\ q_2(t) = q(t + T/3), \\ q_3(t) = q(t + 2T/3), \end{cases}$$

$$\sum_i q_i = 0,\ \sum_i q_i \wedge \dot{q}_i = 0.$$

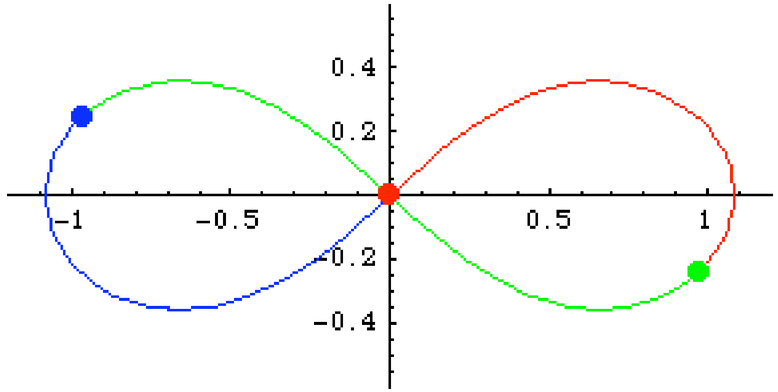


Figure-Eight curve for V_α

$$V_\alpha = \begin{cases} \alpha^{-1} r^\alpha \text{ for } \alpha \neq 0 \\ \log r \text{ for } \alpha = 0 \end{cases}$$

Numerical evidence

Moore: Exist for $\alpha < 2$

CGMS: Exist for $\alpha < 0$ and Stable $\alpha = -1 \pm \epsilon$

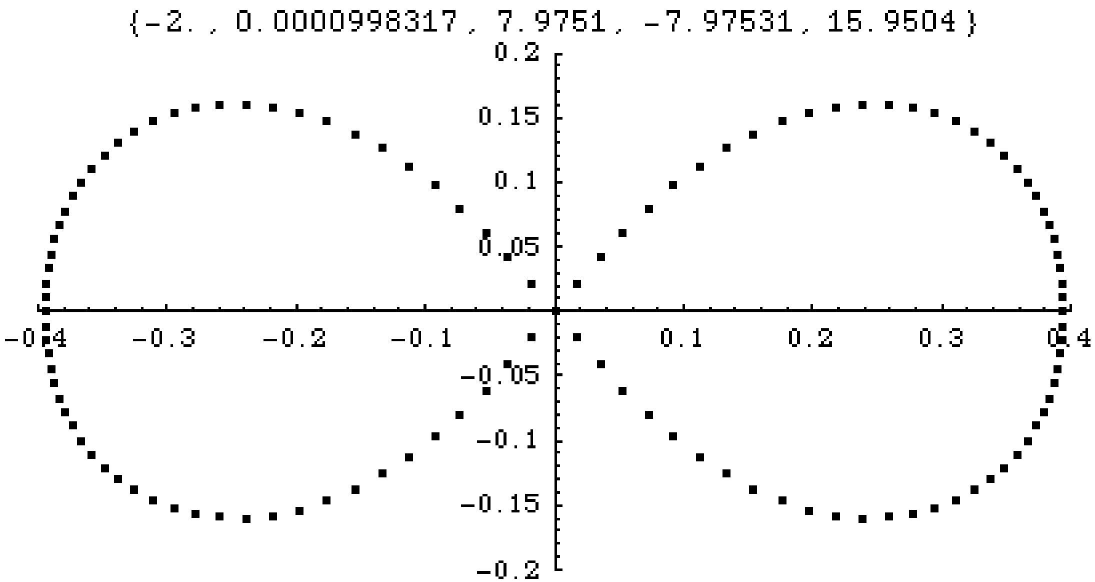


Figure-Eight for $-2 \leq \alpha \leq 1$, $T = 1$.

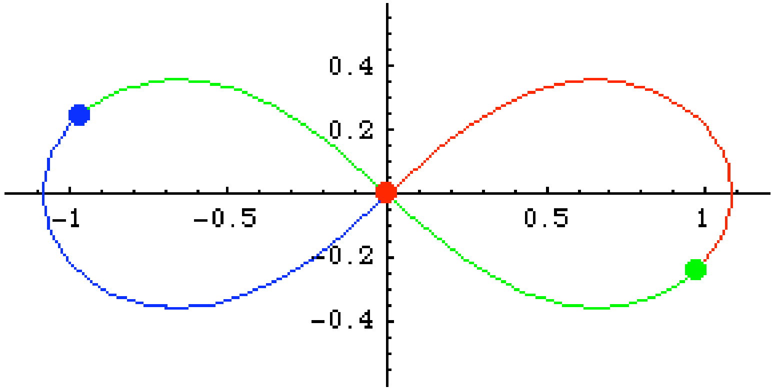
Figure Eight has Zero Angular momentum

Why $L = 0$?

Total angular momentum is conserved.
Therefore,

$$\sum_i q_i \wedge \dot{q}_i = \sum_i < q_i \wedge \dot{q}_i >= 0.$$

$<\bullet>$: time average



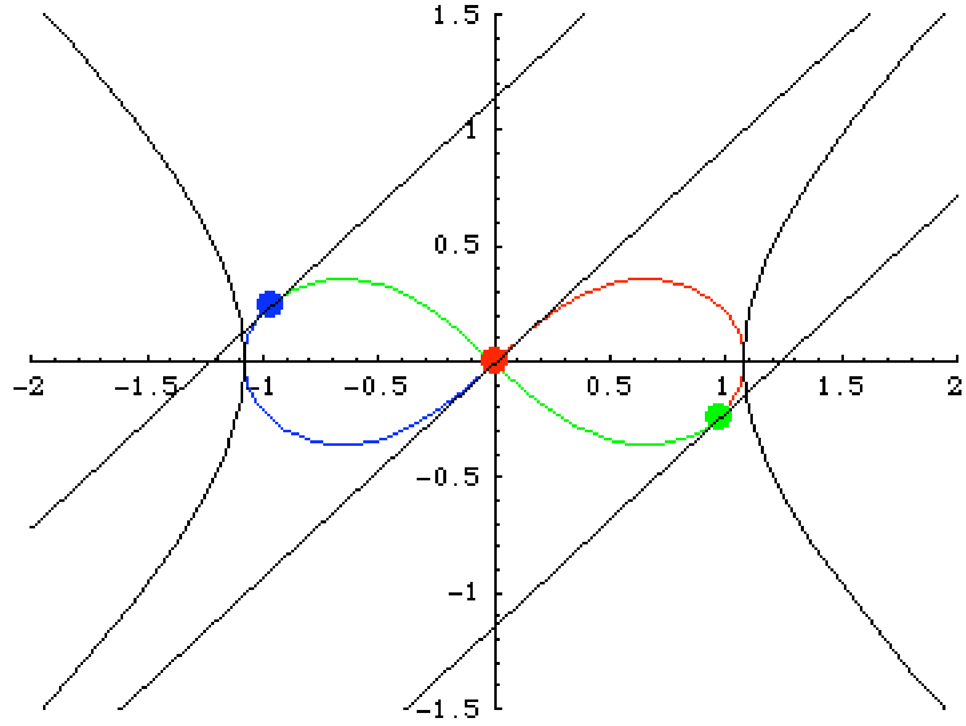
Then, what does $L = 0$ mean?

Three Tangents Theorem

Theorem (FFO). *In the three body problem, if*

$$\sum_i p_i = 0 \ \textit{and} \ \sum_i q_i \wedge p_i = 0,$$

three tangents lines to three bodies must meet at one point for each instant.



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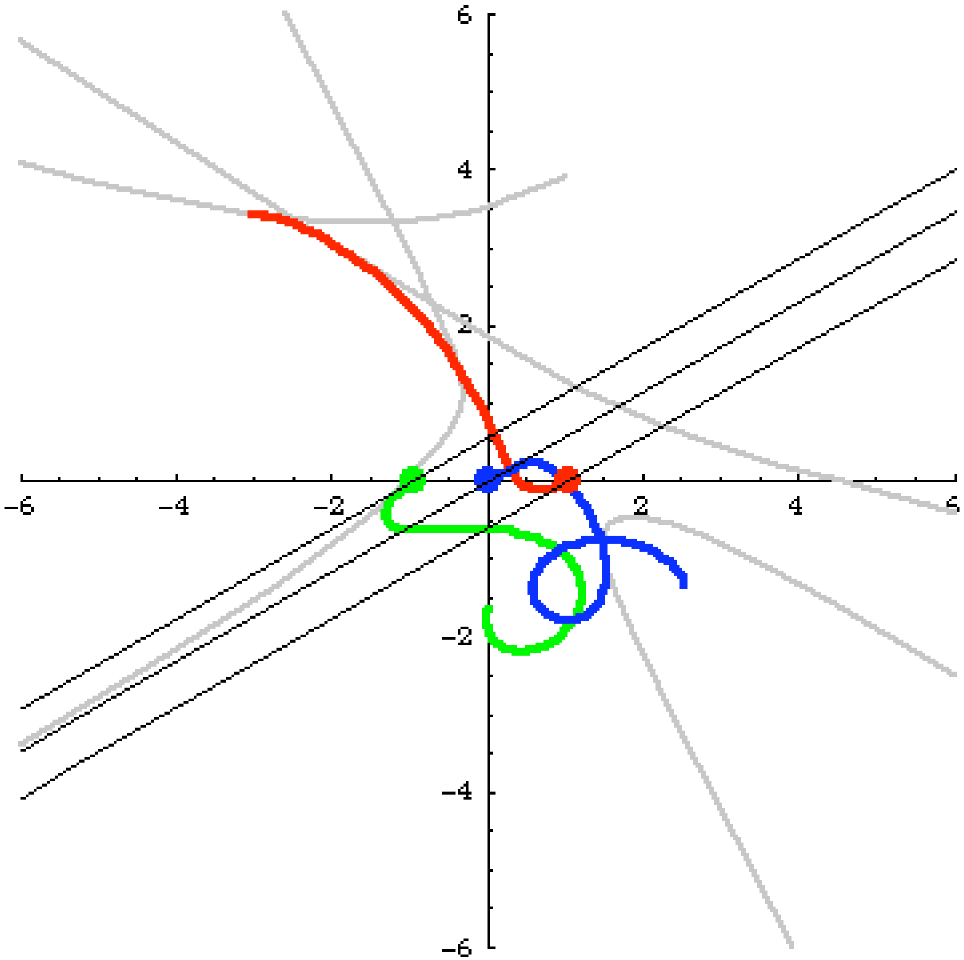


$m_1 = 1.0, m_2 = 1.1, m_3 = 1.2$

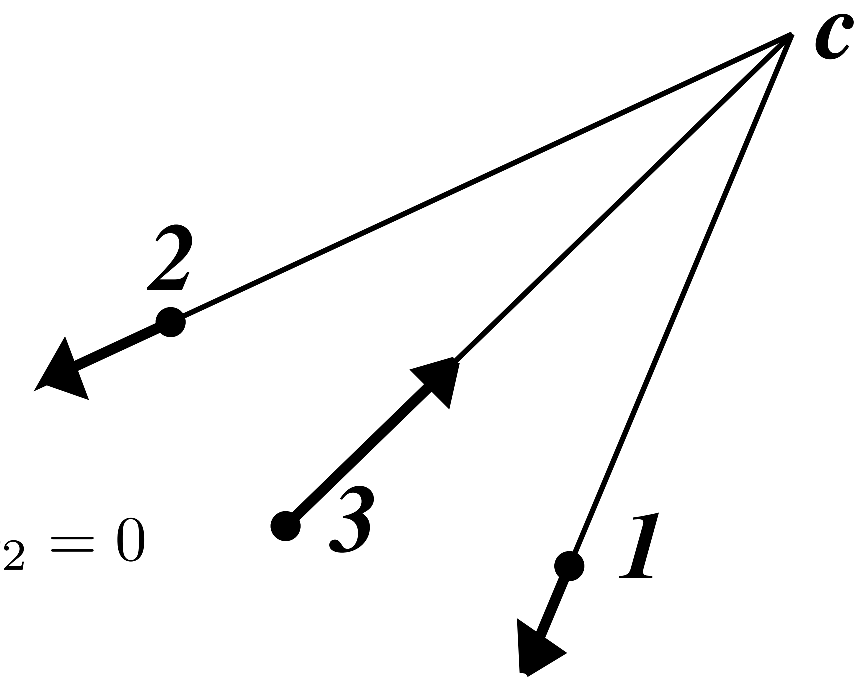
Three Tangents Theorem

$$\sum_i p_i = 0, \sum_i q_i \wedge p_i = 0$$

$$\Rightarrow \sum_i (q_i - C) \wedge p_i = 0.$$

Then,
$(q_1 - C) \wedge p_1 = 0$ and $(q_2 - C) \wedge p_2 = 0$

$$\Rightarrow (q_3 - C) \wedge p_3 = 0.$$

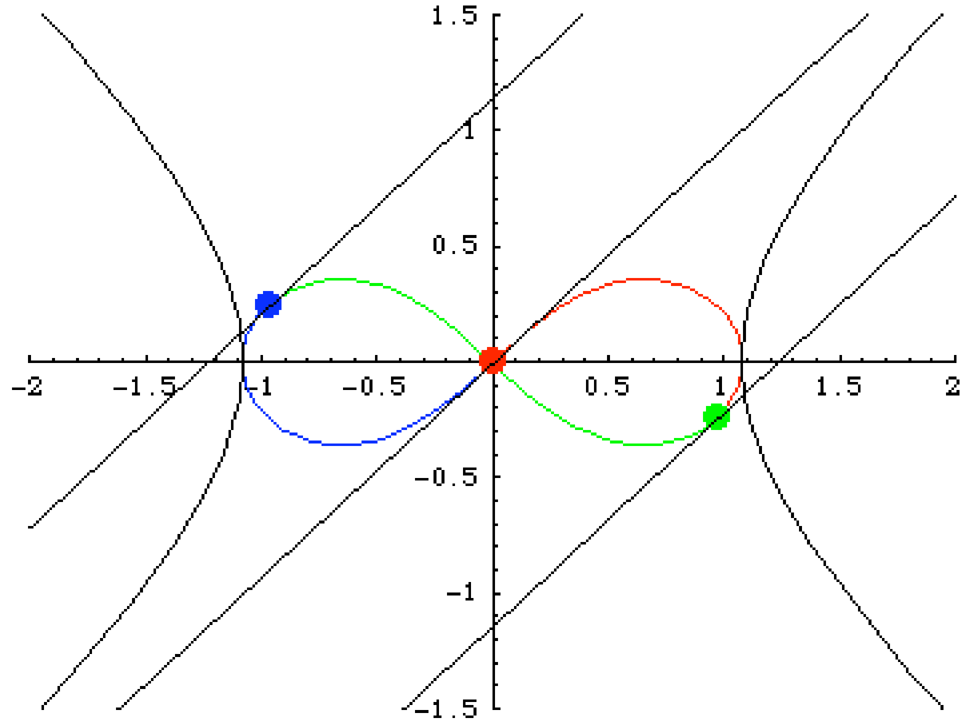


$$C = -\frac{(q_i \wedge p_i)p_j - (q_j \wedge p_j)p_i}{p_i \wedge p_j}$$

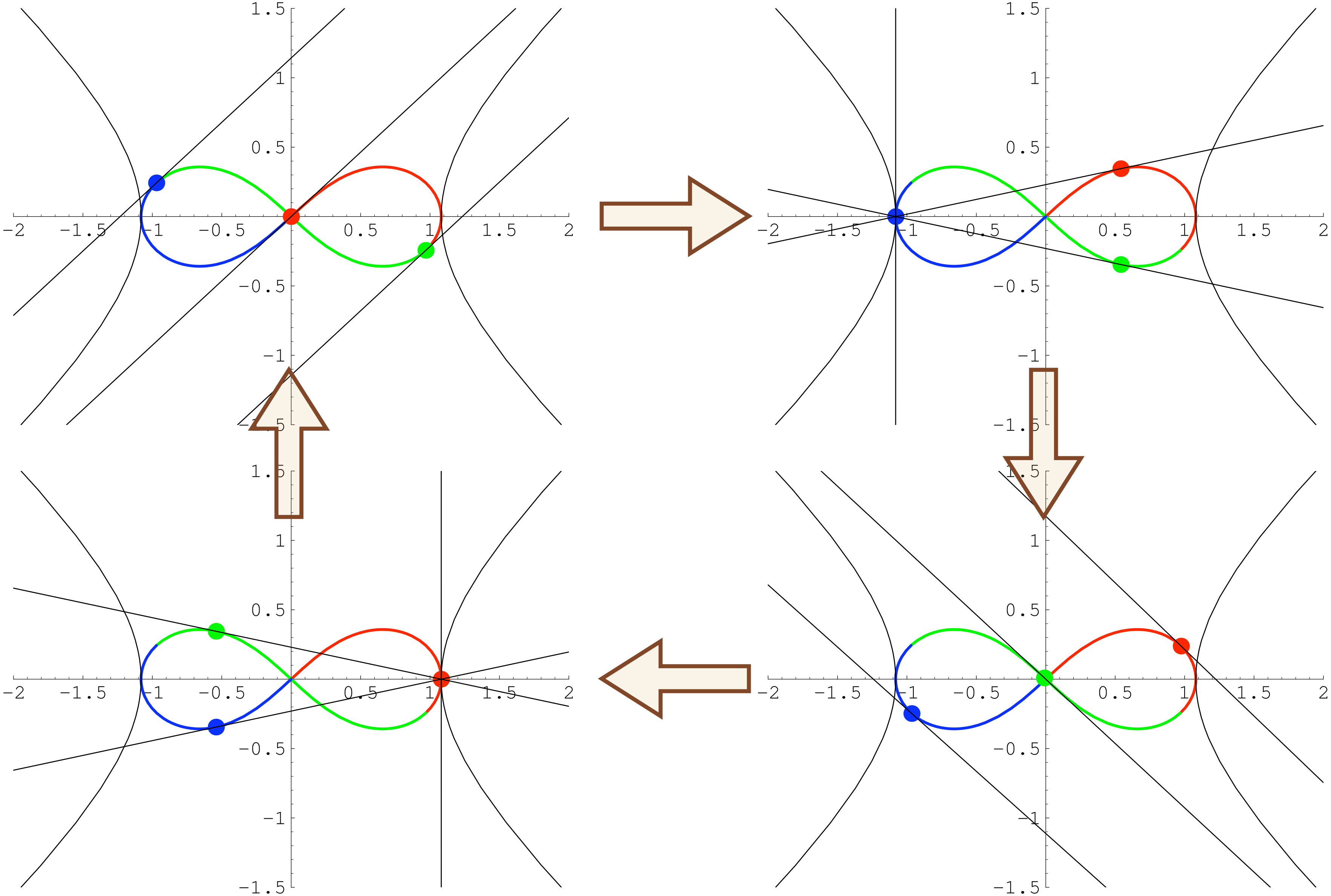
"Center of Velocity"

Three Tangents Theorem

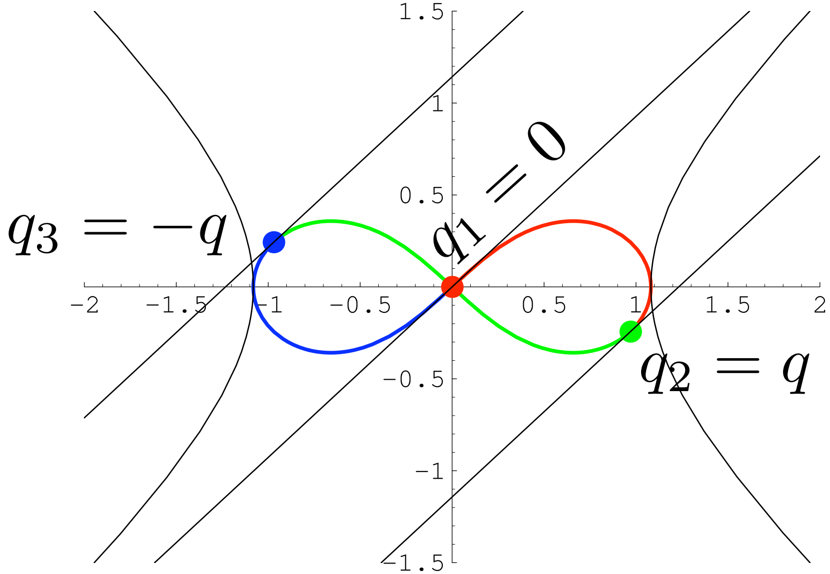
- Shape of the orbit of Figure Eight *x(t)* and the orbit *C(t)* are still unknown.
- Three Tangents Theorem gives an information of the shape.
- For example ...



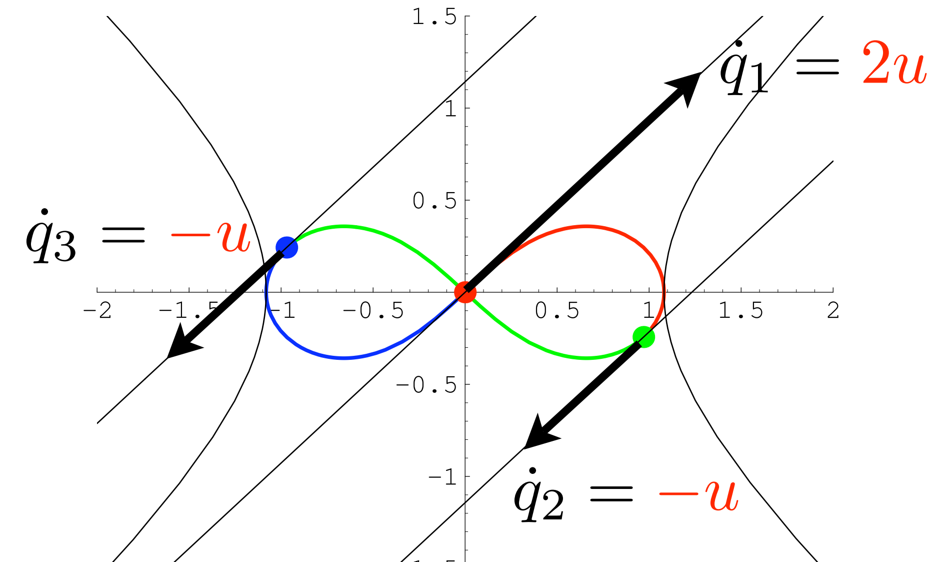
Euler & Isosceles Config.



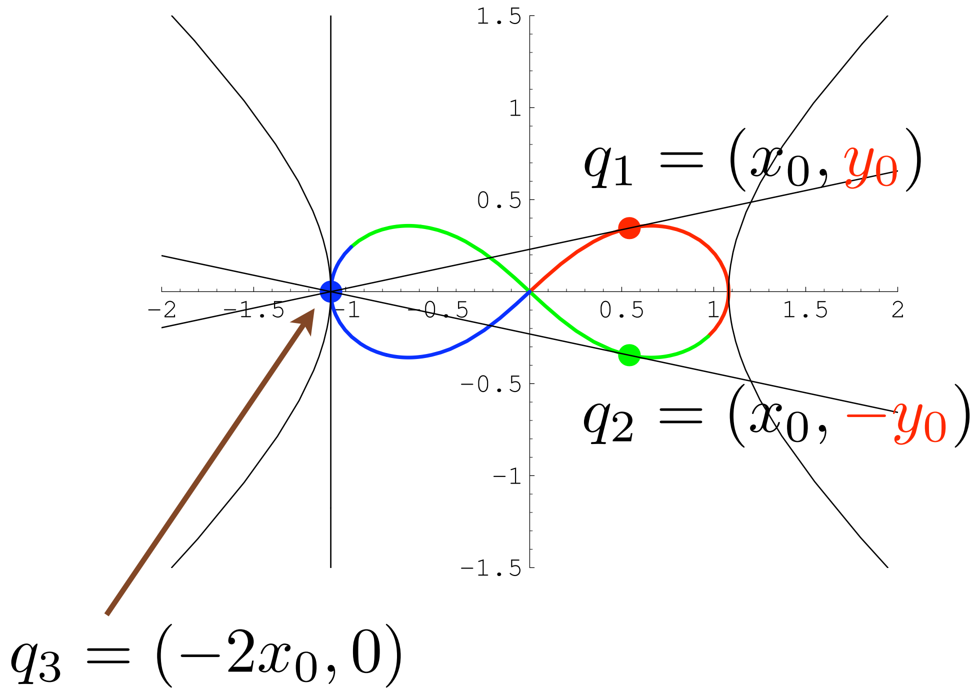
Euler Config.



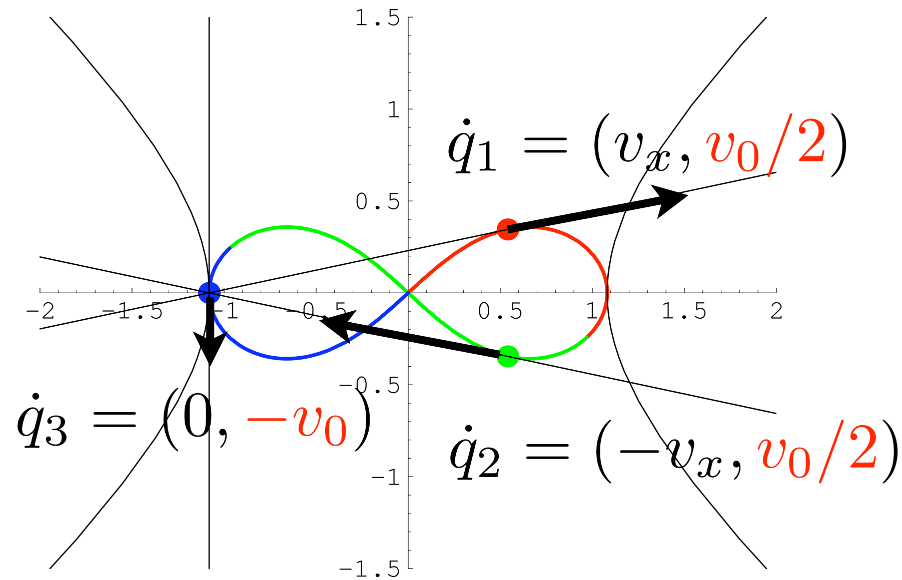
Scale and rotation parameter q
and
two parameters $u = (u_x, u_y)$



Isosceles Config.

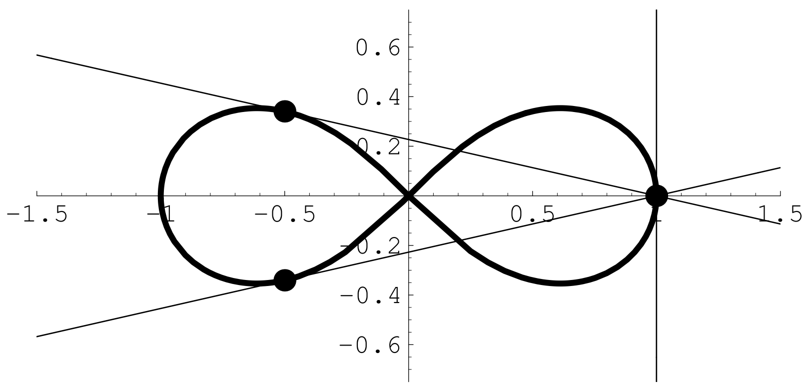


Scale parameter x_0 and two parameters y_0, v_0.

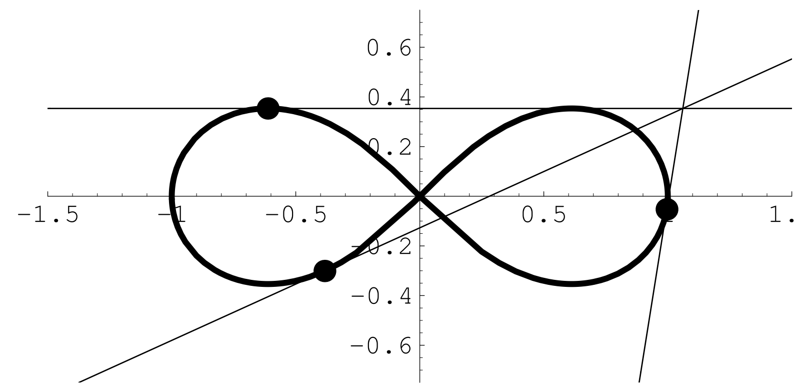


Simplest Curve:
Forth order polynomial

$$x^4 + \alpha x^2 y^2 + \beta y^4 = x^2 - y^2$$



$$\alpha = 2$$



$$\beta = 1$$

Candidate:

Lemniscate $(x^2 + y^2)^2 = x^2 - y^2$

and its scale transform $x \to \mu x, y \to \nu y$

Three Body Choreography on the Lemniscate

Choreograpgy on the Lemniscate

$$q(t)=\left(\frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)},\frac{\mathrm{sn}(t)\mathrm{cn}(t)}{1+\mathrm{cn}^2(t)}\right)\ \text{with } k^2=\frac{2+\sqrt{3}}{4},$$

$$\begin{cases} q_1(t)=q(t),\\ q_2(t)=q(t+T/3),\\ q_3(t)=q(t+2T/3),\end{cases}$$

satisfies the equation of motion $\ddot{q}_i=-\dfrac{\partial}{\partial q_i}U$ with

$$U=\sum_{i<j}\left(\frac{1}{2}\ln r_{ij}-\frac{\sqrt{3}}{24}r_{ij}^2\right).$$

Lemniscate

$$q(t) = \left(\frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}, \frac{\mathrm{sn}(t)\mathrm{cn}(t)}{1+\mathrm{cn}^2(t)} \right) = (x, y)$$

$$\Downarrow$$

$$x^2+y^2 = \frac{\mathrm{sn}^2(1+\mathrm{cn}^2)}{(1+\mathrm{cn}^2)^2} = \frac{\mathrm{sn}^2}{1+\mathrm{cn}^2}, \quad x^2-y^2 = \frac{\mathrm{sn}^2(1-\mathrm{cn}^2)}{(1+\mathrm{cn}^2)^2} = \frac{\mathrm{sn}^4}{(1+\mathrm{cn}^2)^2}$$

$$\Downarrow$$

$$\left(x^2+y^2\right)^2 = x^2 - y^2.$$

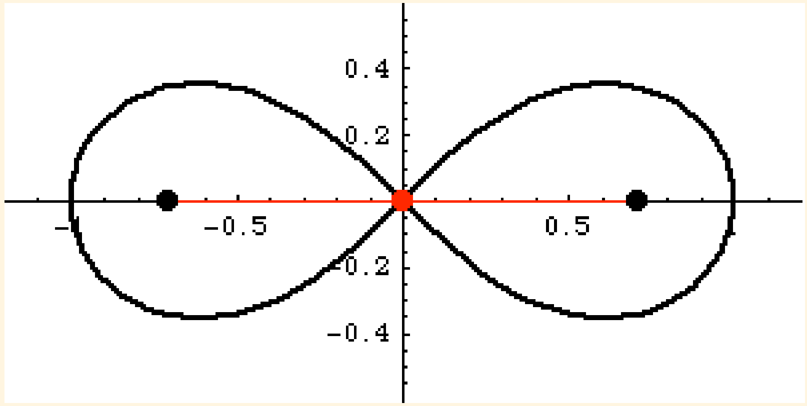
$$\Downarrow$$

$$z = x + iy \Rightarrow z^2 z^{*2} - \frac{1}{2}\left(z^2 + z^{*2}\right)^2 = 0$$

$$\Downarrow$$

$$\left(z^2 - \frac{1}{2}\right)\left(z^{*2} - \frac{1}{2}\right) = \frac{1}{4}$$

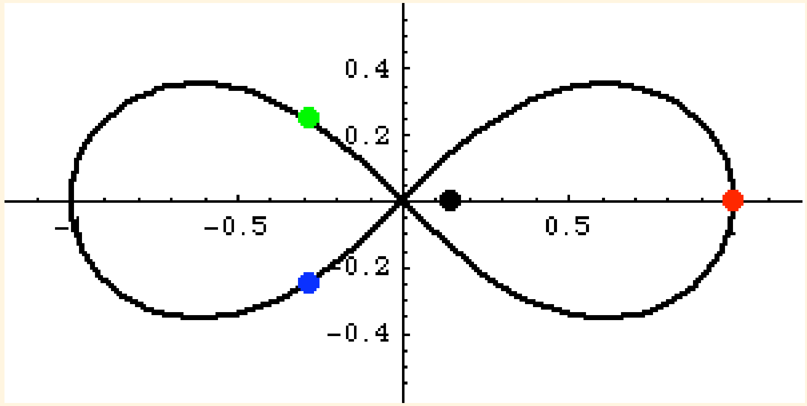
$\Rightarrow$



Center of Mass

$$T = 4K(k)$$

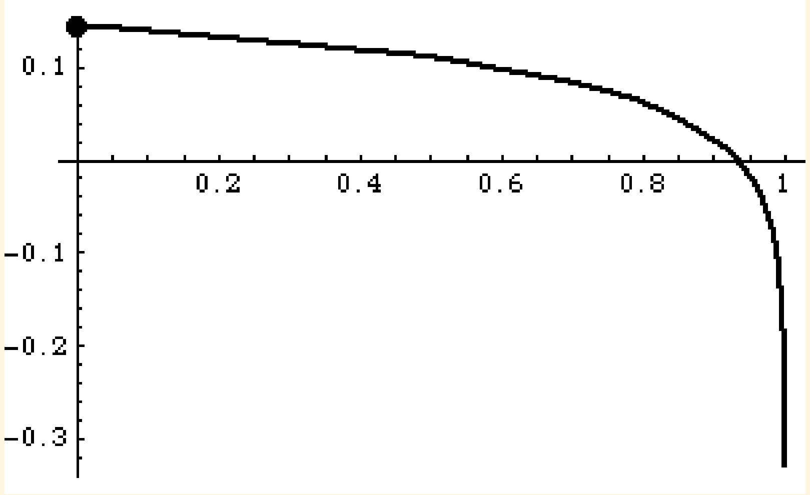
$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$



$$q\left(\frac{T}{4}\right) + q\left(\frac{T}{4}+\frac{T}{3}\right) + q\left(\frac{T}{4}-\frac{T}{3}\right)$$

$$= 0$$

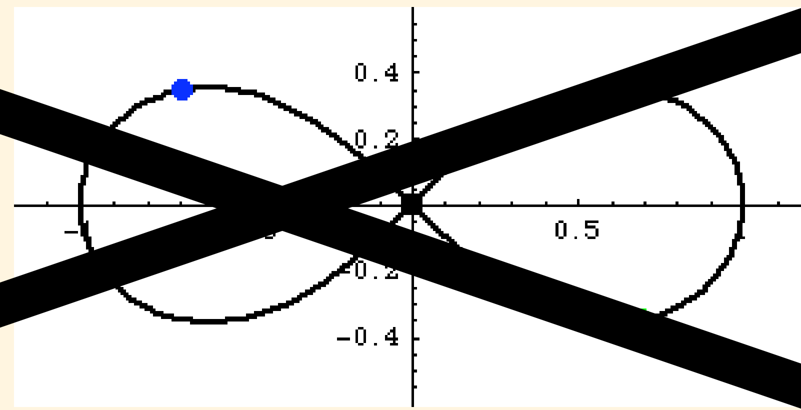
$$\Leftrightarrow k^2 = \frac{2+\sqrt{3}}{4} = 0.9330127\ldots$$



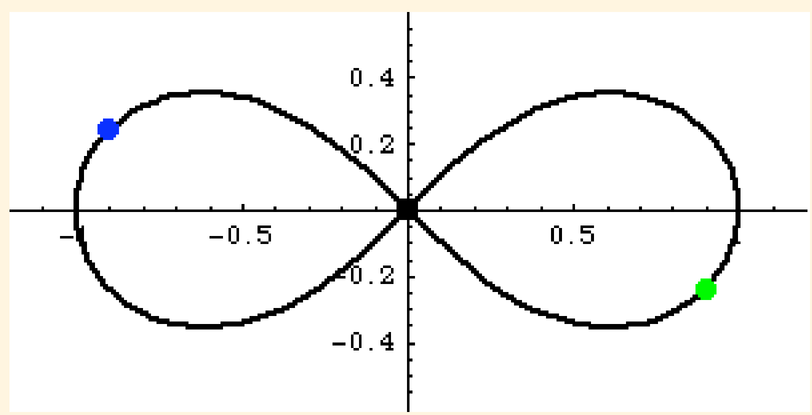
Modulus k^2 vs CM.

Center of Mass stays at the Origin

$$k^2 = 0$$



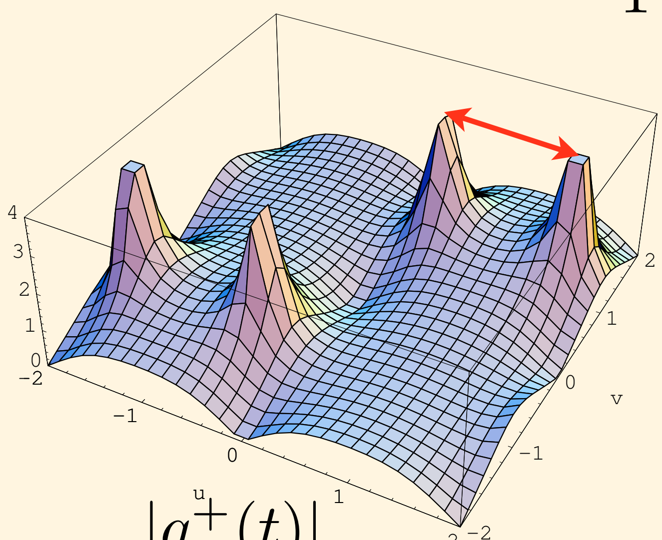
$$k^2 = \frac{2+\sqrt{3}}{4}$$



Center of Mass stays at the Origin

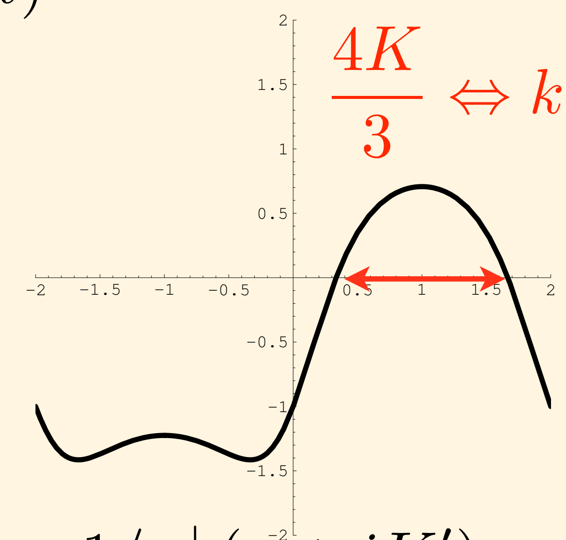
Proof: $q(t) = \left(\dfrac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}, \dfrac{\mathrm{sn}(t)\mathrm{cn}(t)}{1+\mathrm{cn}^2(t)} \right) = (x(t), y(t))$

$$\Rightarrow q^{\pm}(t) = x \pm iy = \frac{\mathrm{sn}(t)}{1 \mp i\mathrm{cn}(t)}$$



$|q^{+}(t)|$,
for $-2K \le \Re t \le 2K$,
$-2K' \le \Im t \le 2K'$

$$\frac{4K}{3} \Leftrightarrow k^2 = \frac{2+\sqrt{3}}{4}$$



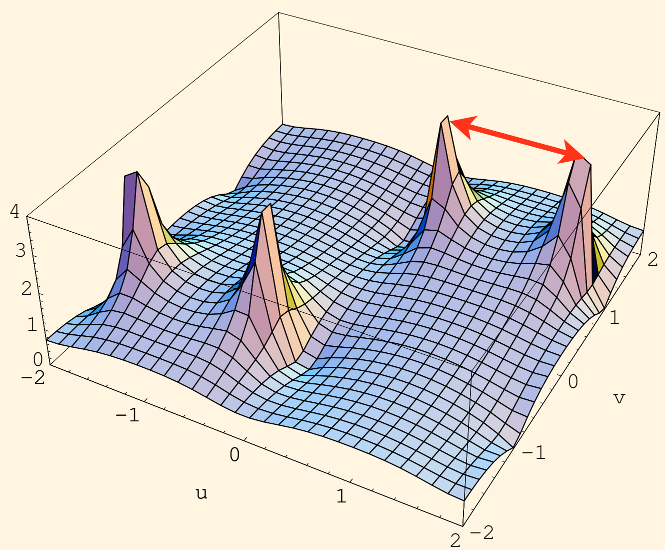
$1/q^{+}(u+iK')$,
for $-2K' \le u \le 2K'$

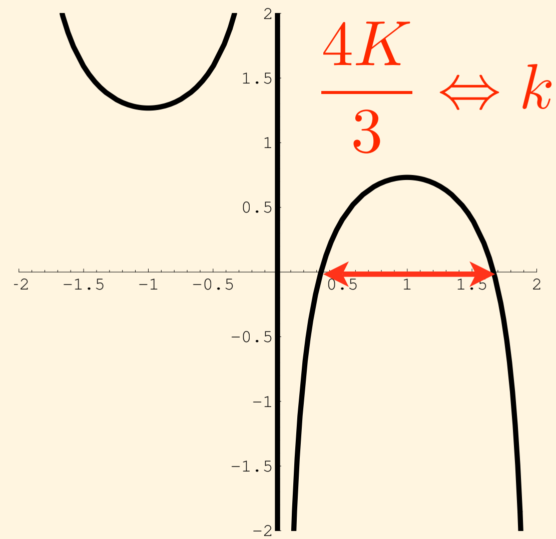
$$\Rightarrow q^{+}(t) + q^{+}\left(t + \frac{4K}{3}\right) + q^{+}\left(t - \frac{4K}{3}\right) = 0$$

Moment of Inertia and Angular Momentum

$$j^+(t) = q^-(t)\dot{q}^+(t) = (x\dot{x} + y\dot{y}) + i\,(x\dot{y} - y\dot{x})$$

$$= \frac{d}{dt}\frac{1}{1 - i\,\mathrm{cn}(t)}$$





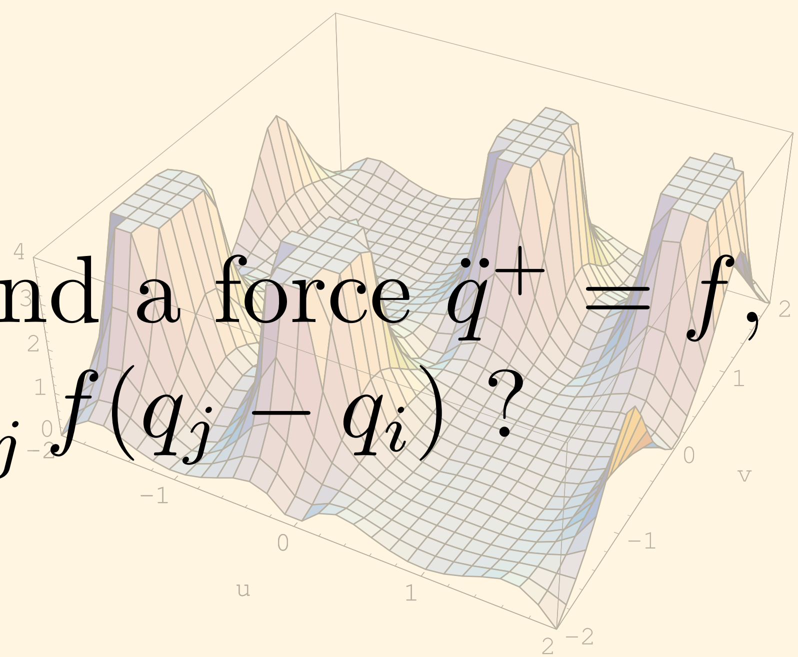
$$\frac{4K}{3} \Leftrightarrow k^2 = \frac{2+\sqrt{3}}{4}$$

$$\sum_i j_i(t) = 0 \Leftrightarrow \sum_i q_i^2 = \text{ const.}, \ \sum_i q_i \wedge \dot{q}_i = 0$$

Equation of Motion

$$q^+ = \frac{\mathrm{sn}(t)}{1 - i\mathrm{cn}(t)} \text{ has simple poles.}$$

$$\Rightarrow \ddot{q}^+ \text{ has triple poles.}$$



Can we find a force $\ddot{q}^+ = f$, that gives $\ddot{q}_i = \sum_{i<j} f(q_j - q_i)$?


N
NE
E
SE
S
SW
W
NW
0
10
20
20
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
190
200
210
220
230
240
250
260
270
280
290
300
310
320
330
340
350

Equation of Motion

We finally find,

$$\ddot{q}^+(t) = \frac{1}{2}\left\{\frac{1}{q^-\left(t+\frac{4K}{3}\right)-q^-(t)}+\frac{1}{q^-\left(t-\frac{4K}{3}\right)-q^-(t)}\right\}$$
$$-\frac{\sqrt{3}}{12}\left\{\left(q^+\left(t+\frac{4K}{3}\right)-q^+(t)\right)+\left(q^+\left(t-\frac{4K}{3}\right)-q^+(t)\right)\right\}.$$

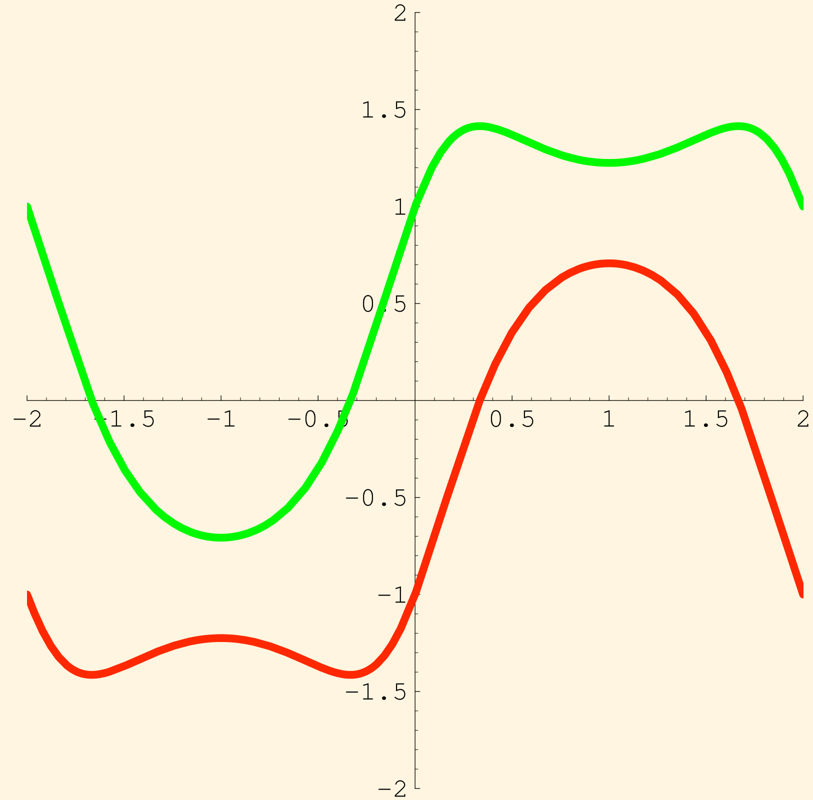
That is

$$\ddot{q}_i = \sum_{j\neq i}\left\{\frac{1}{2}\frac{q_j-q_i}{|q_j-q_i|^2}-\frac{\sqrt{3}}{12}(q_j-q_i)\right\}$$

repulsive

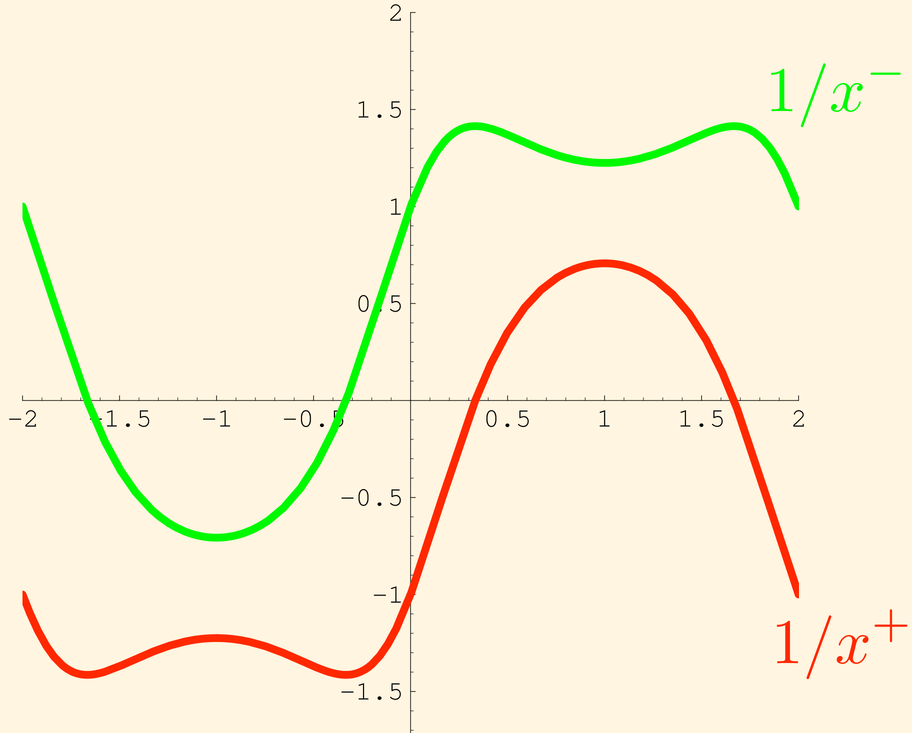
$$\Rightarrow U = \sum_{i<j}\left(\frac{1}{2}\ln r_{ij}-\frac{\sqrt{3}}{24}r_{ij}^2\right).$$

Equation of Motion
Structure of Zeros and Poles



$$\color{green}{1/x^-(u+iK')}\color{black}{,}\ \color{red}{1/x^+(u+iK')}\color{black}{,}$$

Equation of Motion
Structure of Zeros and Poles

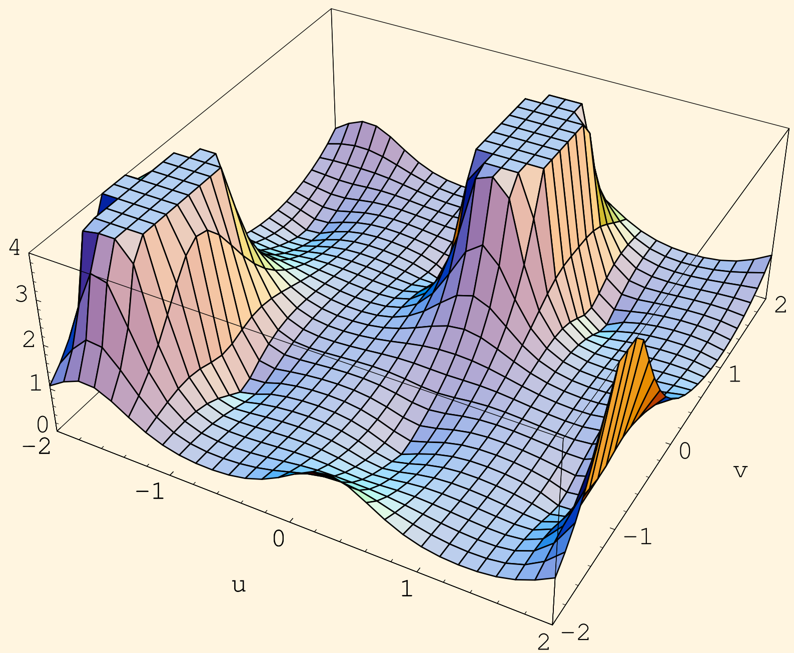


Lemniscate

$$\left({\color{red}x^+}(t)^2 - 1/2\right)\left({\color{green}x^-}(t)^2 - 1/2\right) = 1/4$$

$${\color{red}x^+}(t_0+\varepsilon) = \frac{1}{\varepsilon} + \cdots \Rightarrow {\color{green}x^-}(t_0+\varepsilon)^2 = \frac{1}{2} + \frac{\varepsilon^2}{4} + O(\varepsilon^3)$$

Equation of Motion
Structure of Zeros and Poles

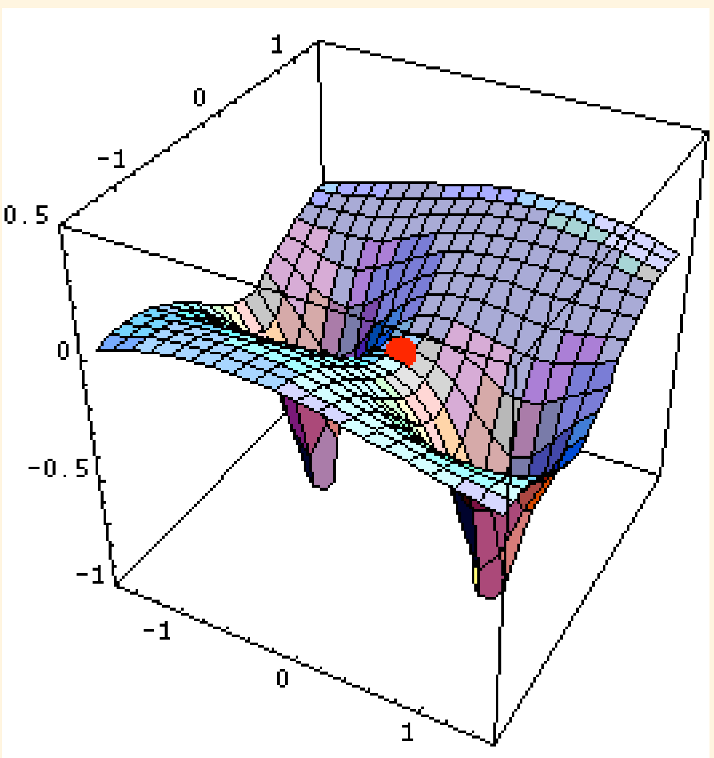


$$\left| \frac{1}{x^-(t+4K/3) - x^-(t)} \right|$$

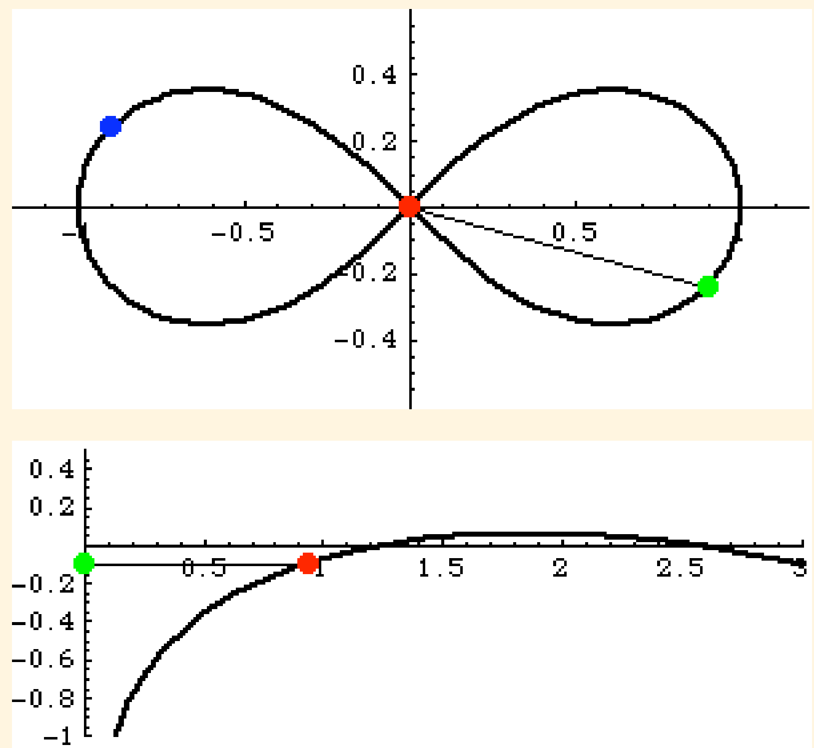
Thus we find the Equation of Motion

Potential Energy

$$V = \sum_{i<j} V_{ij}, \;\; V_{ij} = \frac{1}{2}\ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2.$$

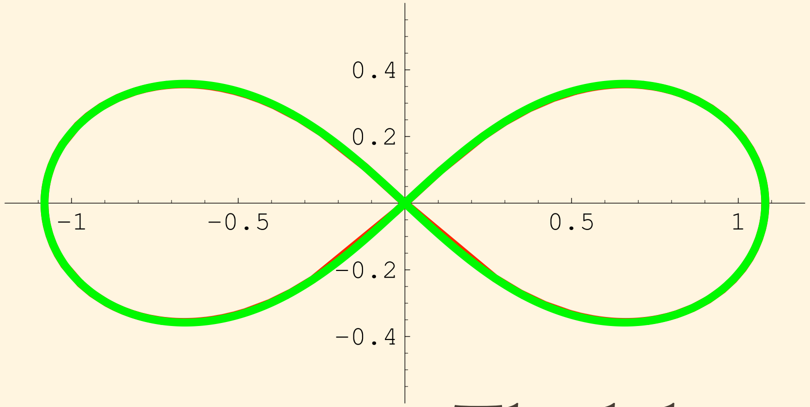


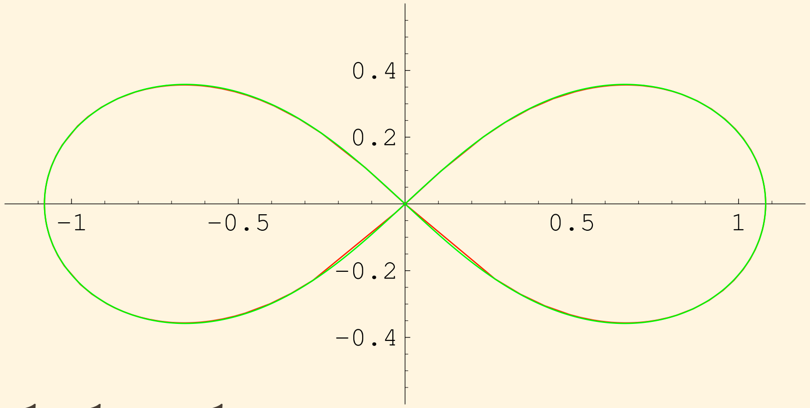
$V_{12} + V_{13}$



V_{12}

Lemniscate & the Figure-Eight





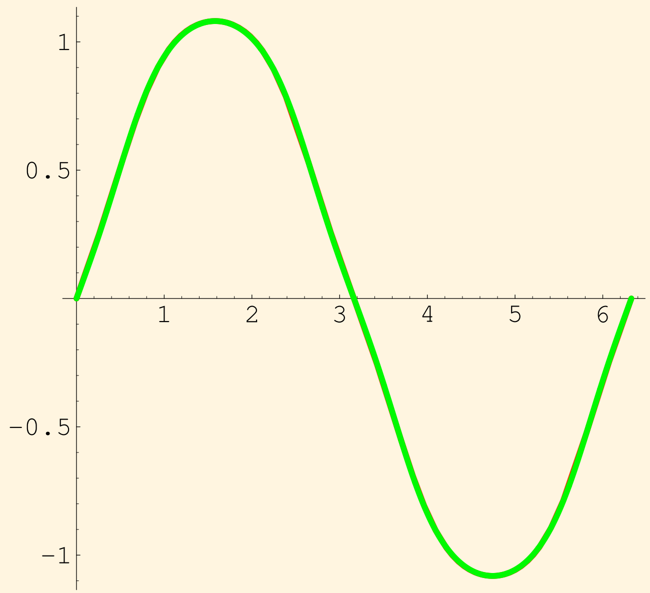
Thick line and thin line

Figure-Eight

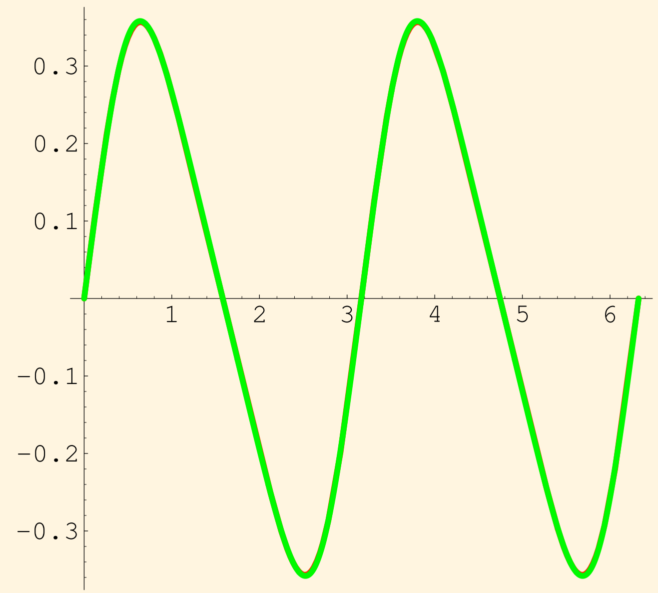
Scaled Leminiscate: $x_{\max} \dfrac{\mathrm{sn}(\tau)}{1+\mathrm{cn}^2(\tau)} \left(1, k^2 \mathrm{cn}(\tau)\right),$

$$k^2 = \frac{2+\sqrt{3}}{4}, \tau = \frac{4K}{T} t$$

Lemniscate & the Figure-Eight



$x(t)$

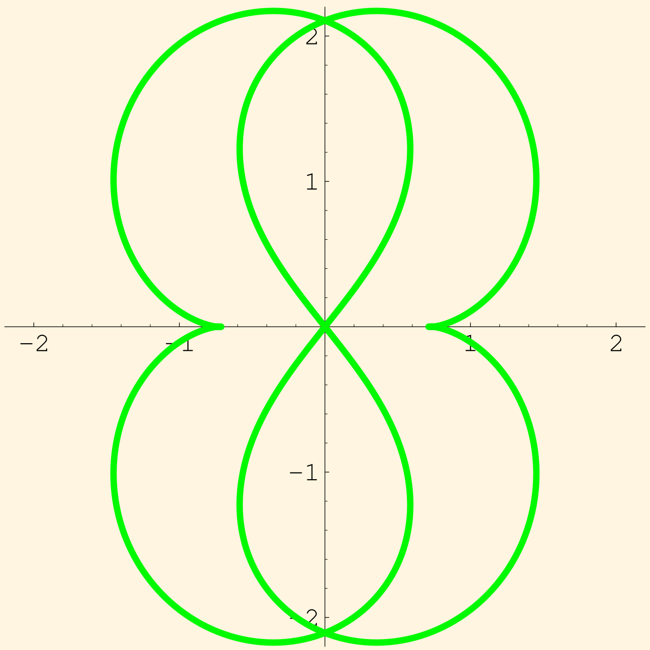


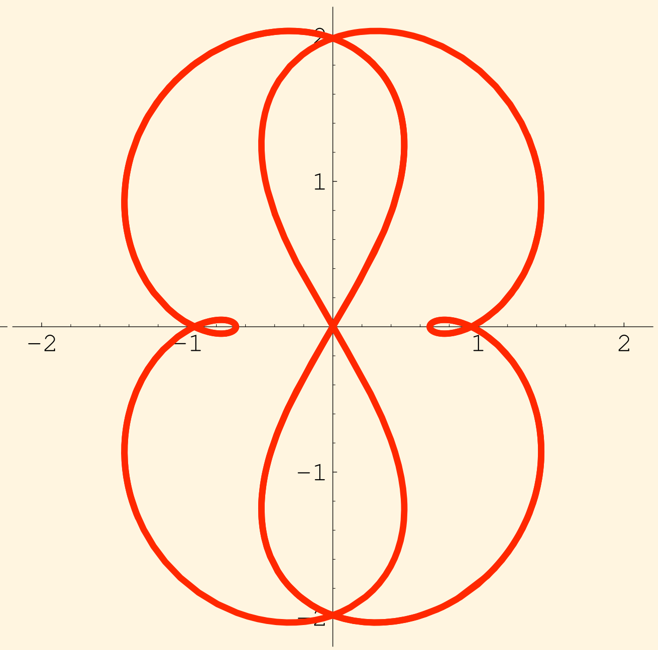
$y(t)$

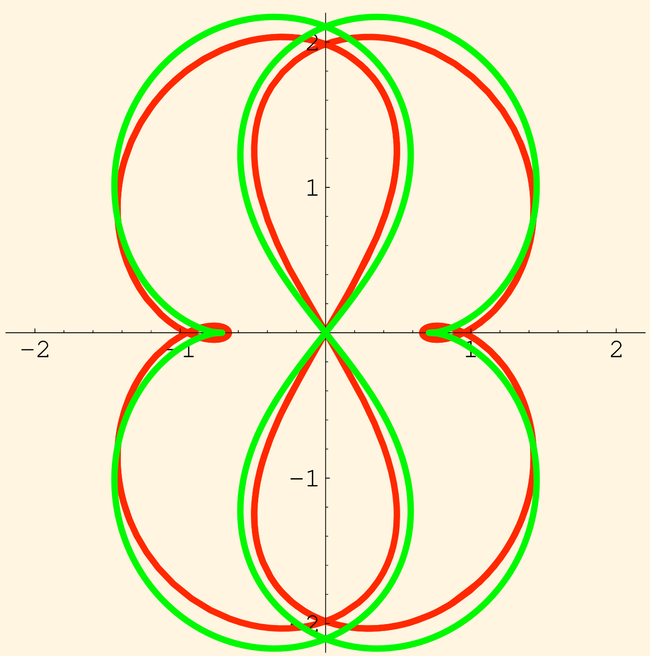
Figure-Eight

Scaled Leminiscate

Lemniscate & the Figure-Eight







$$\ddot{q}(t) = (\ddot{x}, \ddot{y})$$

Figure-Eight

Scaled Leminiscate

Constants on the lemniscate orbit

$$\sum_i q_i = 0,\ \sum_i q_i \wedge \dot{q}_i = 0$$

$$\sum_i q_i^2 = \sqrt{3},\ \sum_i \rho_i^{-2} = 9\sqrt{3},\ \sum_i \dot{q}_i^2 = \frac{3}{4}$$

$$\sum_{i<j} r_{ij}^2 = 3\sqrt{3},\ r_{12}^2 r_{23}^2 r_{31}^2 = \frac{3\sqrt{3}}{2}.$$

$$\because \begin{cases} \rho^{-1} = \dfrac{\dot{q} \wedge \ddot{q}}{|\dot{q}|^3} \Rightarrow \rho^{-2} = 9q^2, \\ \dot{q}^2 + \left(k^2 - \dfrac{1}{2}\right) q^2 = \dfrac{1}{2}, \\ \displaystyle\sum_{i<j} r_{ij}^2 = 3 \sum_i q_i^2. \end{cases}$$

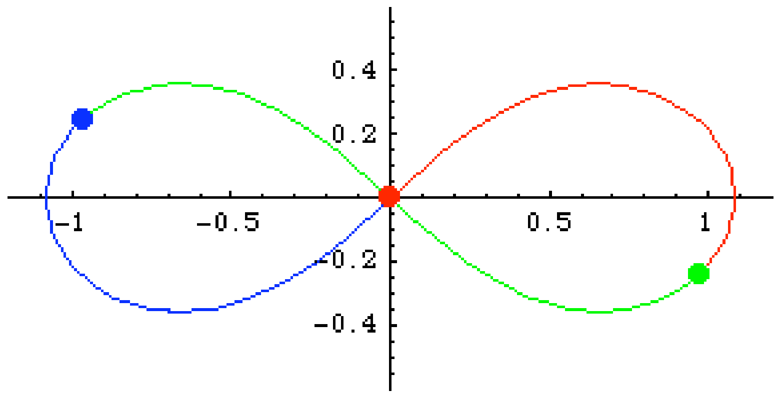
Inconstancy of the Moment of Inertia

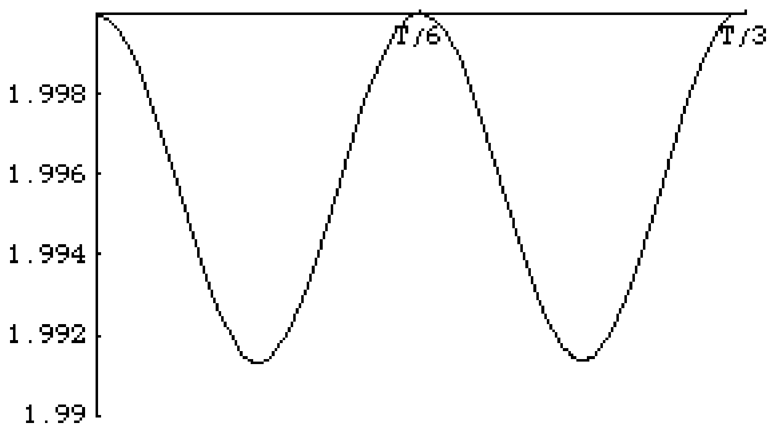
Moment of inertia $I = \sum_i q_i^2$.

Potential Energy

$$U_\alpha = \begin{cases} \alpha^{-1} r^\alpha \text{ for } \alpha \neq 0 \\ \log r \text{ for } \alpha = 0. \end{cases}$$

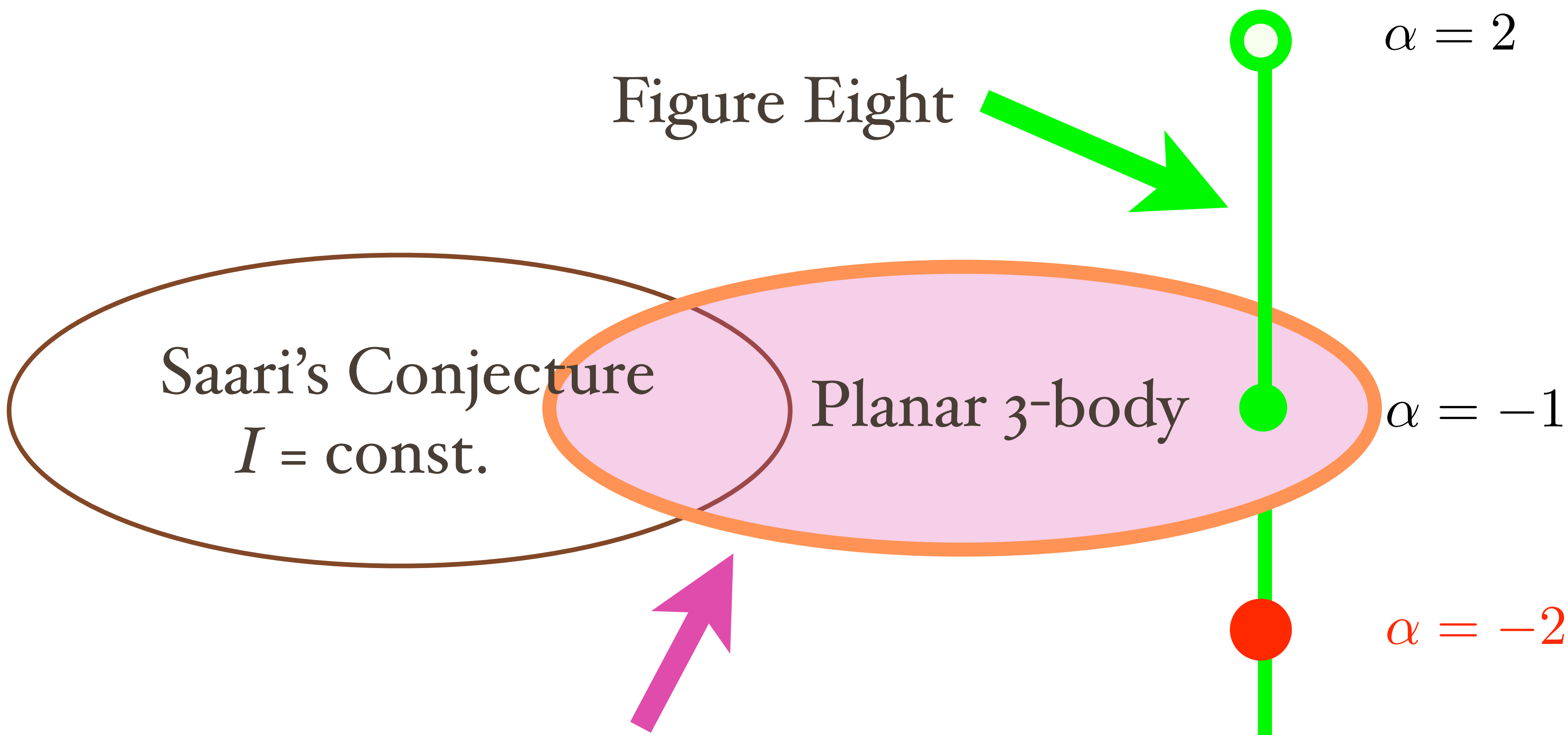
Problem (Chenciner). *Show that the moment of inertia I stays constant if and only if $\alpha = -2$.*





$$\frac{\Delta I}{I} \sim \frac{1}{200} \text{ for } \alpha = -1$$

Saari's Conjecture



C. McCord: equal masses
J. Llibre & E. Pina: equall masses & ...
R. Moeckel: general masses

$$V = \frac{r^\alpha}{\alpha}$$

Lagrange-Jacobi identity

$$I = \sum_i q_i^2,$$

$$K = \frac{1}{2}\sum_i \dot{q}_i^2, \; V_\alpha = \frac{1}{\alpha}\sum_{i<j} r_{ij}^\alpha$$

$$\Rightarrow \frac{d^2 I}{dt^2} = 4K - 2\alpha V_\alpha = 4E - 2(2+\alpha)V_\alpha$$

For $\alpha = -2$,

$$\frac{d^2 I}{dt^2} = E \Rightarrow I = 2Et^2 + c_1 t + c_2.$$

Lagrange-Jacobi identity

$$\frac{d^2 I}{dt^2} = 4K - 2\alpha V_\alpha = 4E - 2(2+\alpha)V_\alpha$$

For $\alpha \neq -2$,

$$I(t) = \text{ const.} \;\Rightarrow \frac{d^{2+n} I}{dt^{2+n}}(0) = -2(2{+}\alpha)\frac{d^n V_\alpha}{dt^n}(0) = 0$$

Infinitely many initial conditions
for Finite degrees of freedom

... with some exceptions.

exception 1
Harmonic Oscillator

$$\frac{d^2I}{dt^2} = 4K - 2\alpha V_\alpha = 4E - 2(2+\alpha)V_\alpha$$

For $\alpha = 2$,

$$V_2 = \frac{1}{2}\sum_{i<j}(q_i - q_j)^2 = \frac{3}{2}\sum_i q_i^2 = \frac{3}{2}I$$

$$(\because \sum_i q_i = 0).$$

$$\therefore \frac{d^2I}{dt^2} = 4E - 3(2+2)I$$

exception 2
One-dim. x^4 Potential

$$\frac{d^2I}{dt^2} = 4K - 2\alpha V_\alpha = 4E - 2(2+\alpha)V_\alpha$$

For one-dimentional x^4 potential,

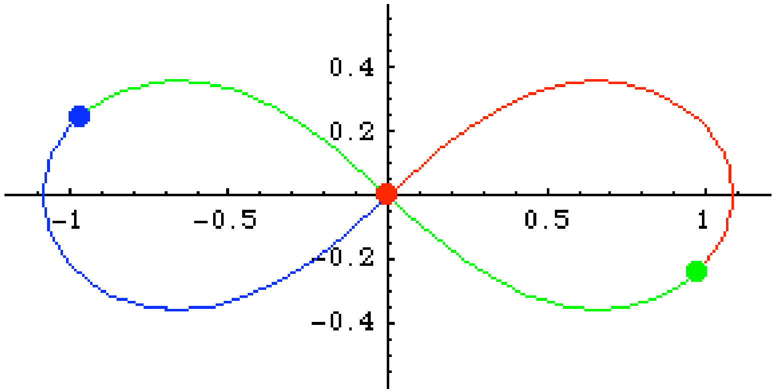
$$V_4 = \sum_{i<j}(x_i - x_j)^4 = \frac{1}{2}\left(\sum_{i<j}(x_i - x_j)^2\right)^2 = \frac{9}{2}I^2$$

$$\therefore \frac{d^2I}{dt^2} = 4E - 9(2+4)I^2$$

FFO prove ...

We consider motions with conditions

1. Equal masses $m_i = 1$,
2. $\sum_i q_i \wedge \dot{q}_i = 0$,
3. $q_3(0) = 0$,
4. $I =$ const. > 0.



Theorem. *No motion satisfies conditions 1, 2, 3 and 4 except for* $\alpha = -2, 2, 4$.

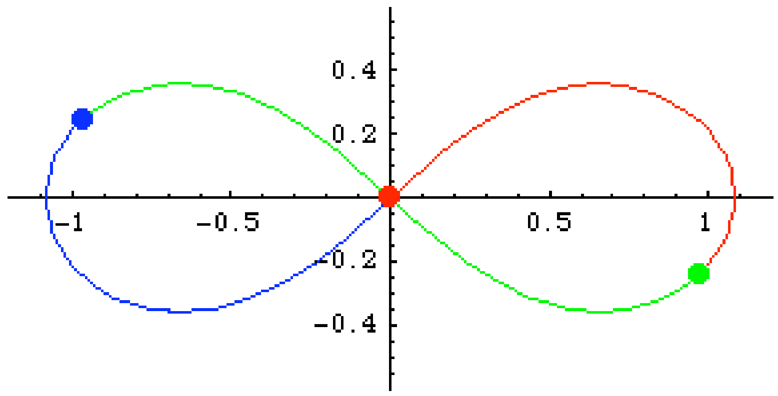
- $\alpha = 2 \Rightarrow$ exception 1. No Eight.
- $\alpha = 4 \Rightarrow$ exception 2. No Eight.

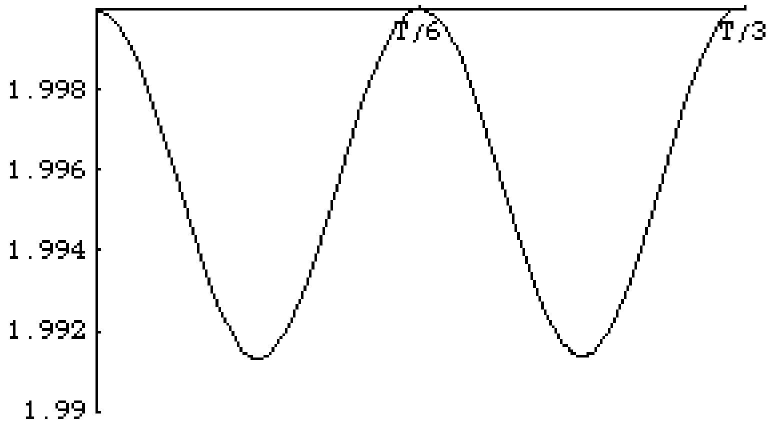
Chenciner's Problem is solved

Moment of inertia $I = \sum_i q_i^2$.

Potential Energy

$$U_\alpha = \begin{cases} \alpha^{-1} r^\alpha \text{ for } \alpha \neq 0 \\ \log r \text{ for } \alpha = 0. \end{cases}$$



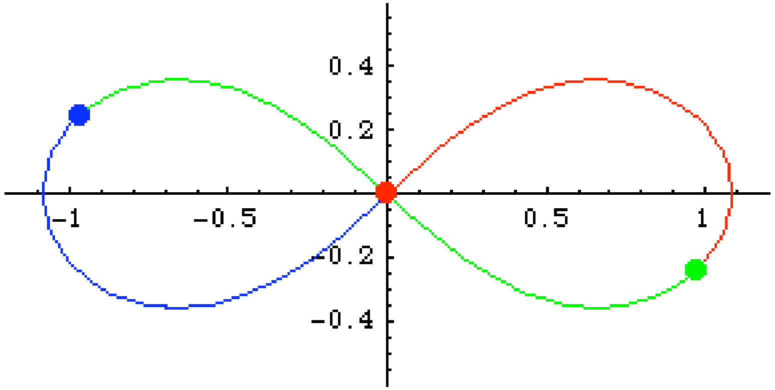


Problem (Chenciner). *Show that the moment of inertia I stays constant if and only if $\alpha = -2$.*

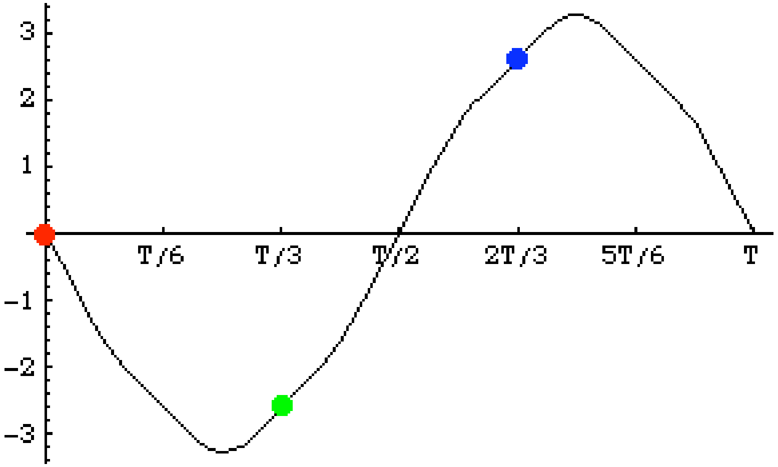
$$\frac{\Delta I}{I} \sim \frac{1}{200} \text{ for } \alpha = -1$$

Convexity of Each Lobe

Theorem (FM). *Each lobe of the eight solution is a convex curve.*



$$\kappa = \frac{\dot{q} \wedge \ddot{q}}{|\dot{q}|^3} = 0 \Leftrightarrow q = 0$$

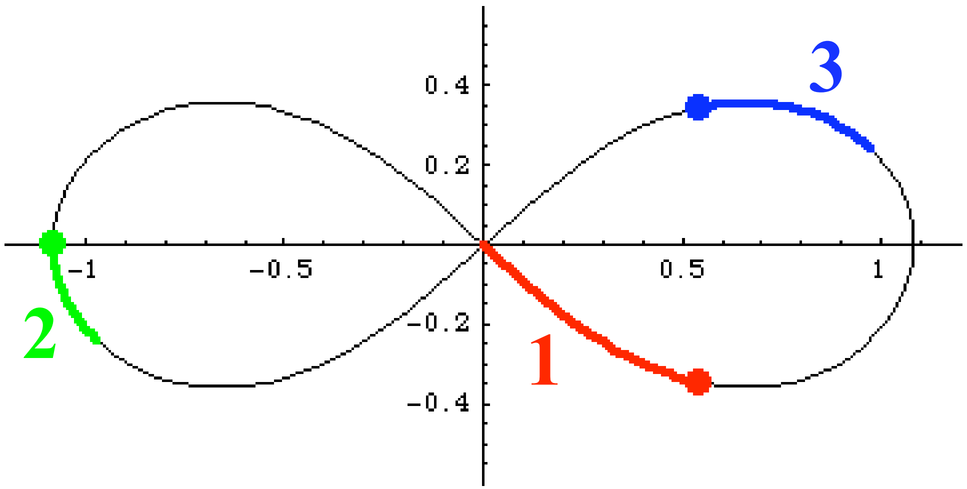


Computer assisted proof:
T. Kapela & P. Zgliczyński

No Isosceles, No Collinear

In this *OPEN* interval

$$\dot{\ell}_1 = \left(\frac{1}{r_{21}^3} - \frac{1}{r_{31}^3} \right) (q_2 \wedge q_3) \neq 0$$



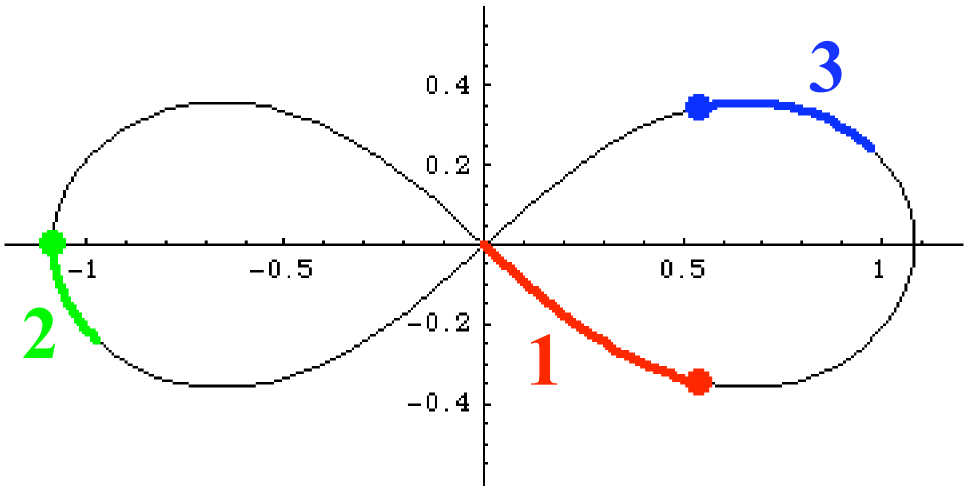
(CM)

$$\Downarrow$$

- $r_{13} < r_{12} < r_{23}$ *distance ordering*
- $q_1 \wedge q_2 = q_2 \wedge q_3 = q_3 \wedge q_1 < 0$

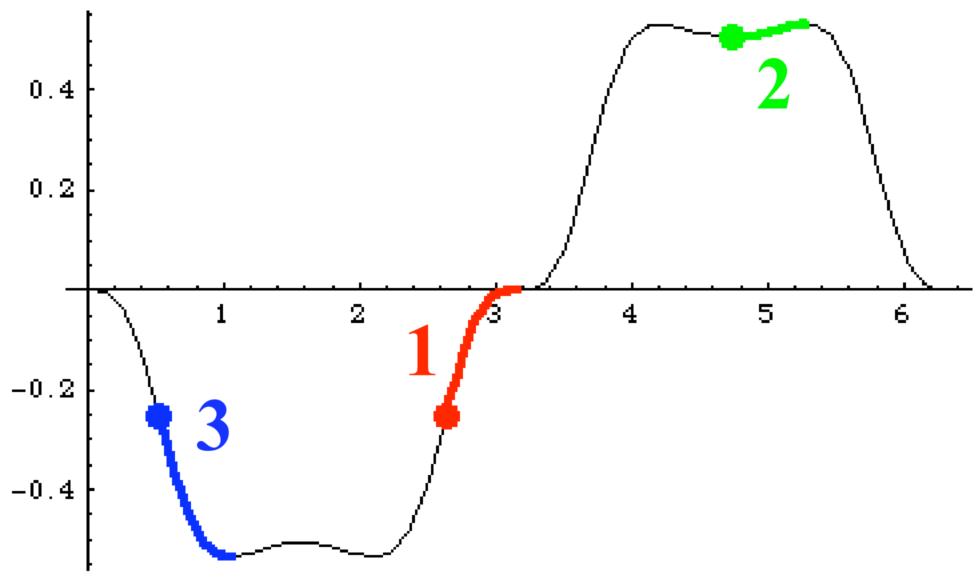
No Isosceles, No Collinear (CM)

- $r_{13} < r_{12} < r_{23}$
- $q_1 \wedge q_2 = q_2 \wedge q_3 = q_3 \wedge q_1 < 0$
- $\dot{\ell}_1 = \left(\frac{1}{r_{21}^3} - \frac{1}{r_{31}^3} \right) (q_2 \wedge q_3)$



$$\Downarrow$$

- $\ell_1 < 0, \dot{\ell}_1 > 0$
- $\ell_2 > 0, \dot{\ell}_2 > 0$
- $\ell_3 < 0, \dot{\ell}_3 < 0$

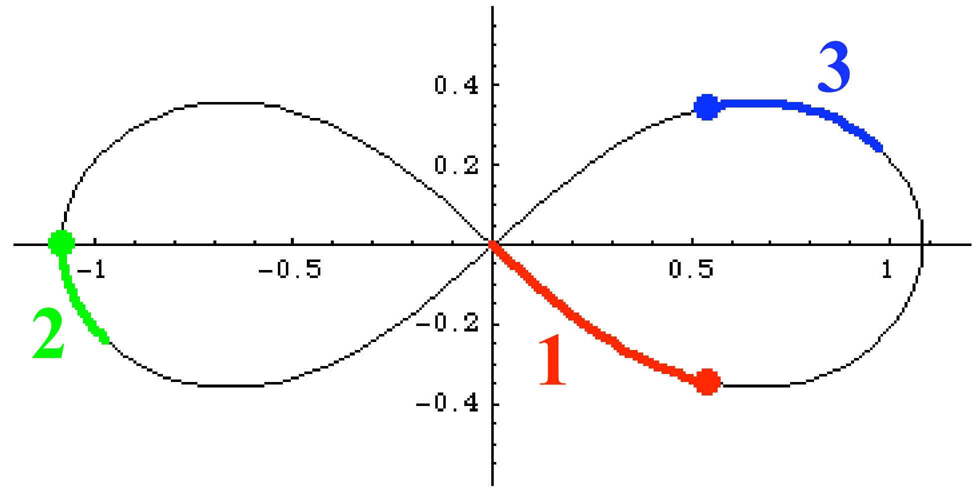


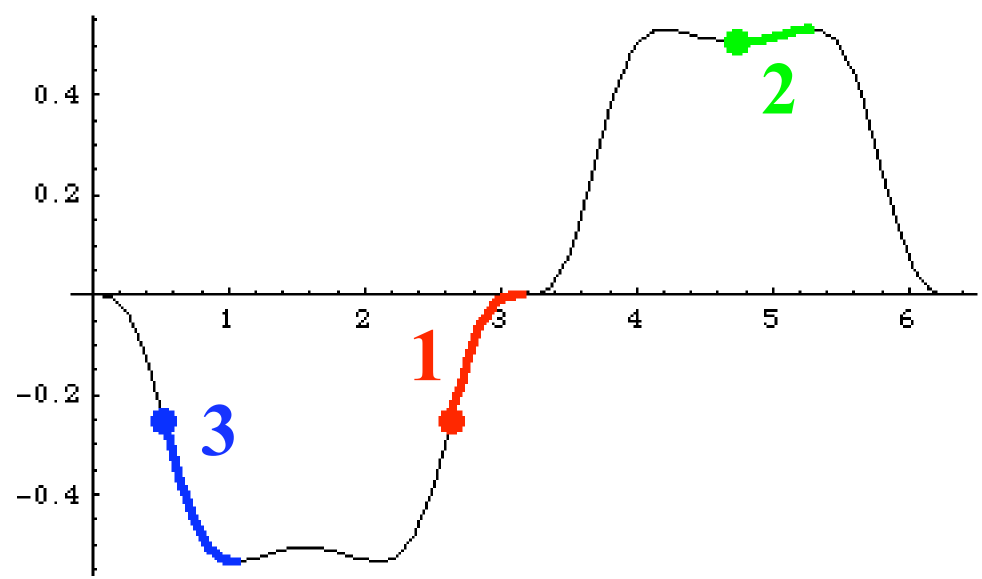
Star Shapedness

Each lobe is star shaped.
(CM)

$$\ell = q \wedge \dot{q} = r^2\dot{\theta} = 0 \Leftrightarrow q = 0.$$

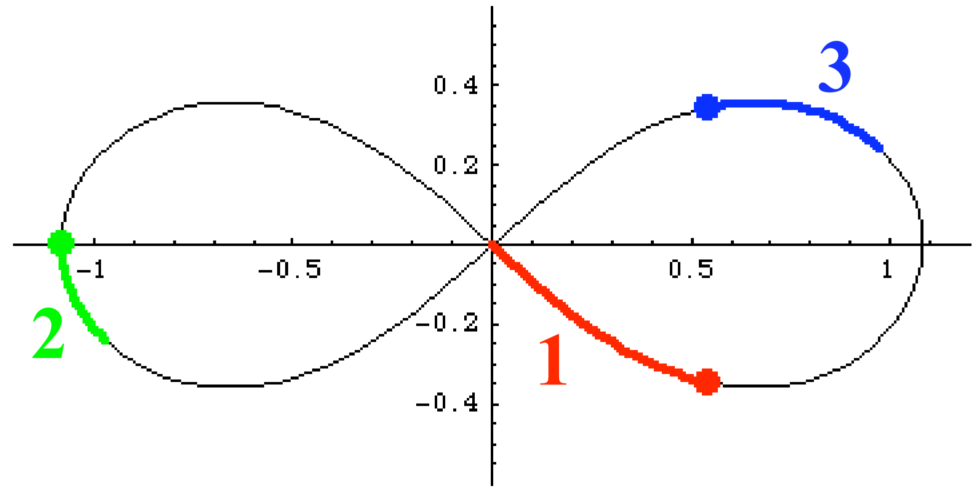
- Right lobe: $\ell < 0,\ \dot{\theta} < 0$
  ($\ell_1 < 0,\ \ \ell_3 < 0$)
- Left lobe: $\ell > 0,\ \dot{\theta} > 0$
  ($\ell_2 > 0$)





To each Mass
Its own Quadrant

Observed by FFO.
Proved by FM.

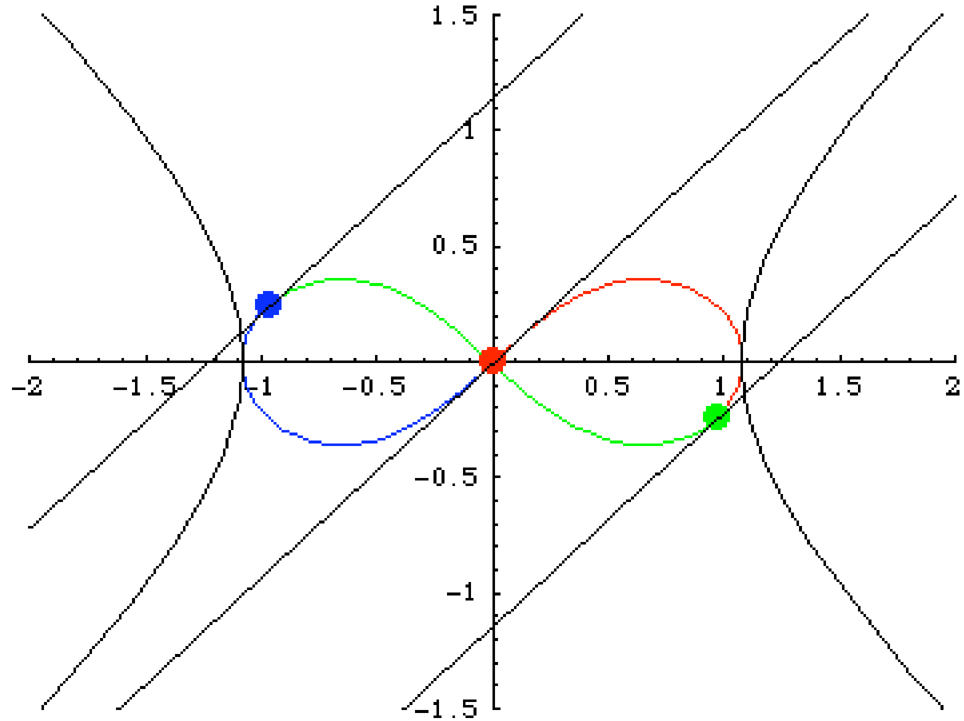


$\because$ Star shapedness,
distance ordering $r_{13} < r_{12} < r_{23}$
and $q_1 + q_2 + q_3 = 0$.

$$\ddot{y}_1 = \frac{y_3 - y_1}{r_{13}^3} + \frac{y_2 - y_1}{r_{12}^3}$$
$$> \frac{y_3 - y_1}{r_{12}^3} + \frac{y_2 - y_1}{r_{12}^3}$$
$$= \frac{-3y_1}{r_{12}^3} > 0$$

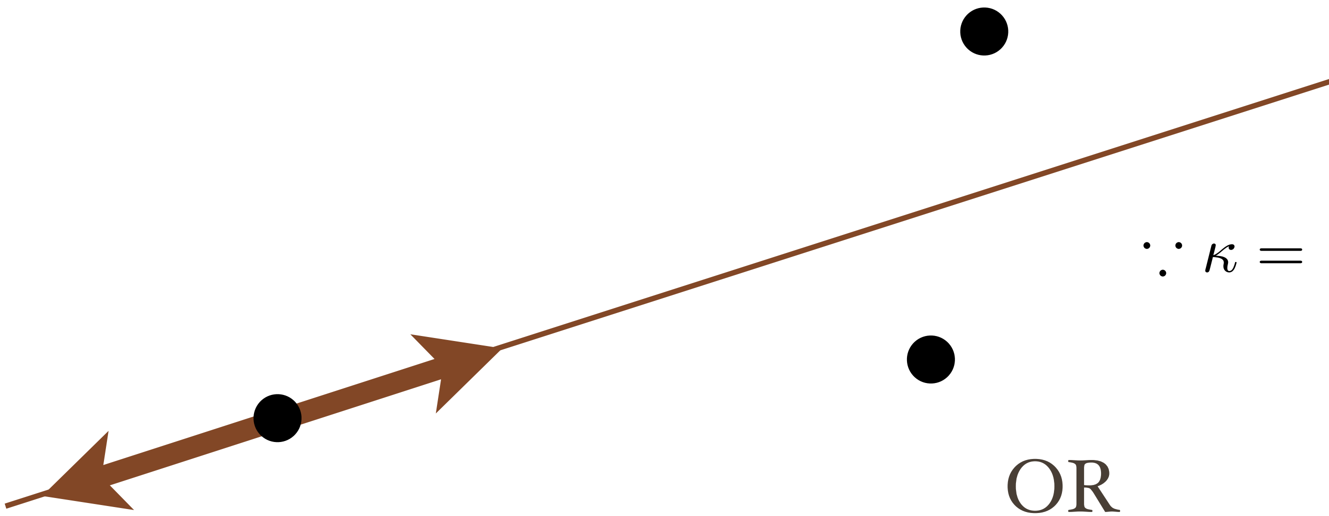
$$\Rightarrow \ddot{y}_1 > 0, \dot{y}_1 > 0, y_1 < 0$$

To each Mass and Center of velocity Its own Quadrant



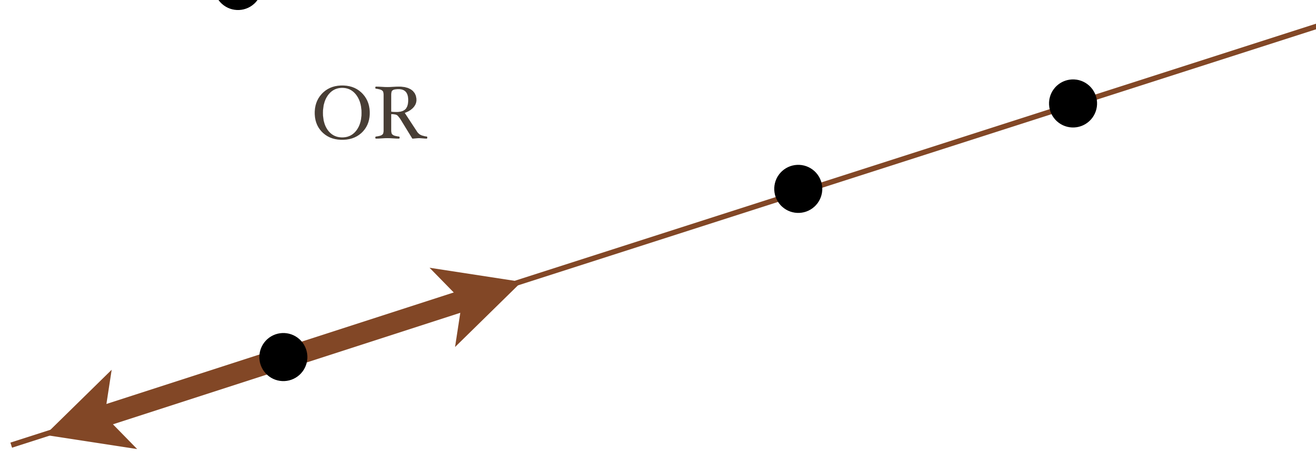
Splitting Lemma

Lemma. *If $\kappa = 0$, tangent line ℓ splits the other two masses or all three masses are on this line.*

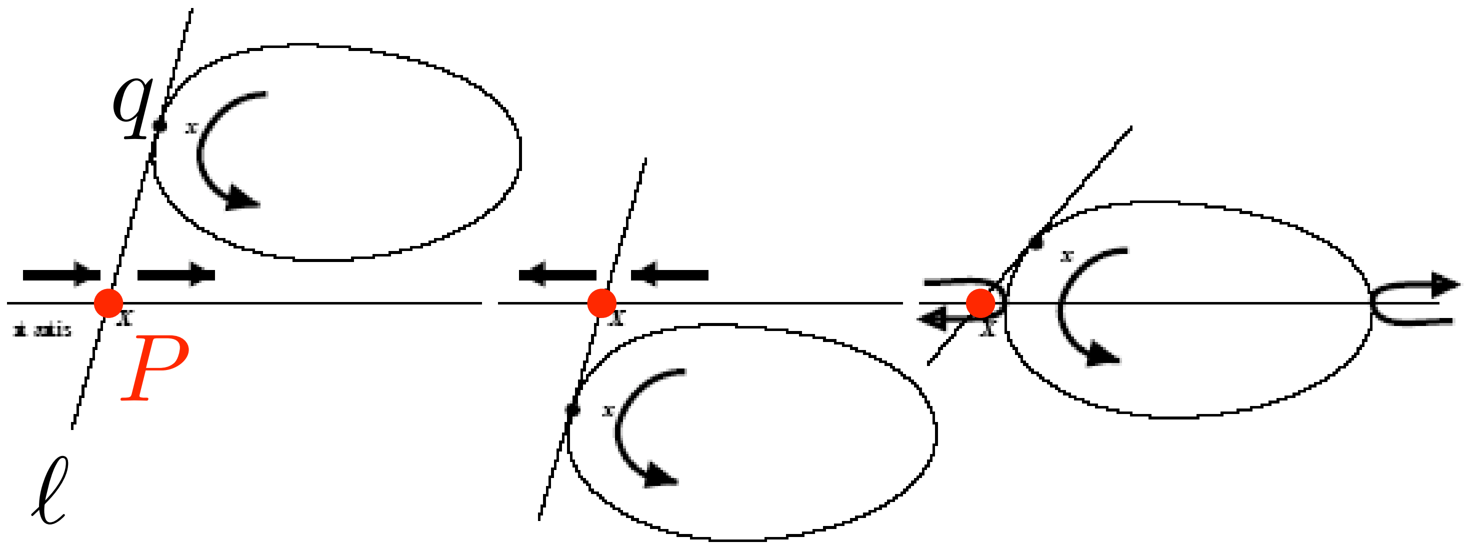


$$\because \kappa = \frac{\dot{q} \wedge \ddot{q}}{|\dot{q}|^3} = 0 \Leftrightarrow \dot{q} \wedge \ddot{q} = 0$$

OR



Convexity Proposition



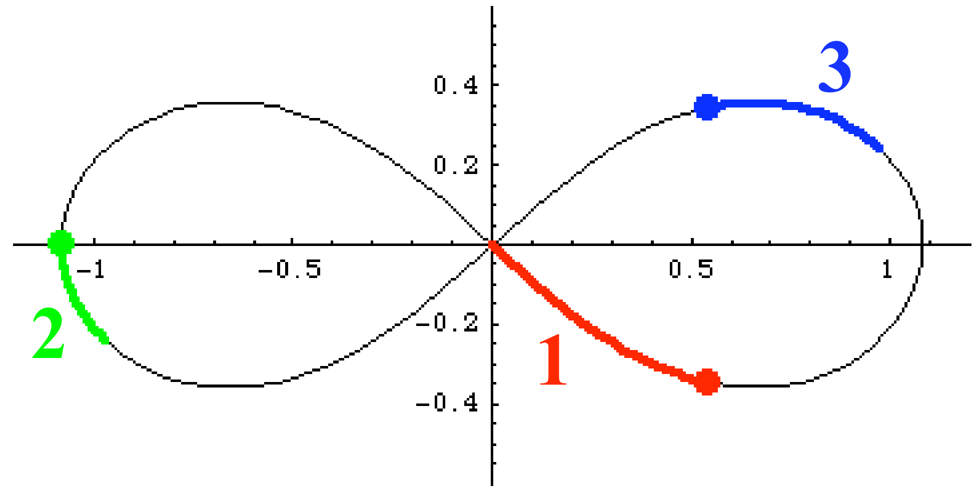
$P(t) = (p(t), 0)$: crossing point of the tangent line ℓ and $y = 0$.

$$p = \frac{q \wedge \dot{q}}{\dot{y}} \Rightarrow \frac{dp}{dt} = \frac{|\dot{q}|^3 \kappa\, y}{\dot{y}^2}.$$

$$\therefore \dot{q} \neq 0, \kappa \neq 0, y \neq 0 \Rightarrow \frac{dp}{dt} \neq 0.$$

Convexity of Arc 1

$$\begin{cases} \ddot{y}_1 > 0, \dot{y}_1 > 0, y_1 < 0, \\ \dot{\ell}_1 = q_1 \wedge \ddot{q}_1 > 0, \\ \ell_1 = q_1 \wedge \dot{q}_1 < 0. \end{cases}$$



Three vectors
q_1, $\dot{q}_1$ and $\ddot{q}_1$ are linear
dependent in 2-dimention.

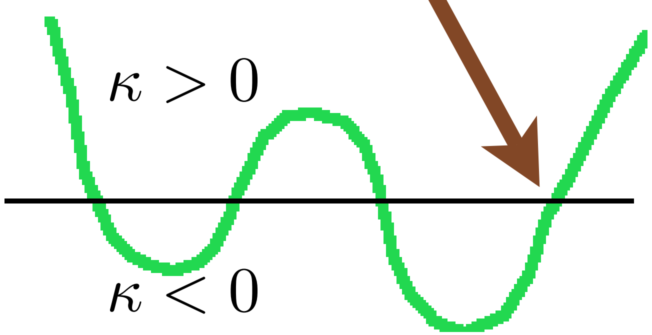
$$0 = y_1(\dot{q}_1 \wedge \ddot{q}_1) + \dot{y}_1(\ddot{q}_1 \wedge q_1) + \ddot{y}_1(q_1 \wedge \dot{q}_1)$$
$$= y_1|\dot{q}_1|^3\kappa_1 - \dot{y}_1\dot{\ell}_1 + \ddot{y}_1\ell_1$$

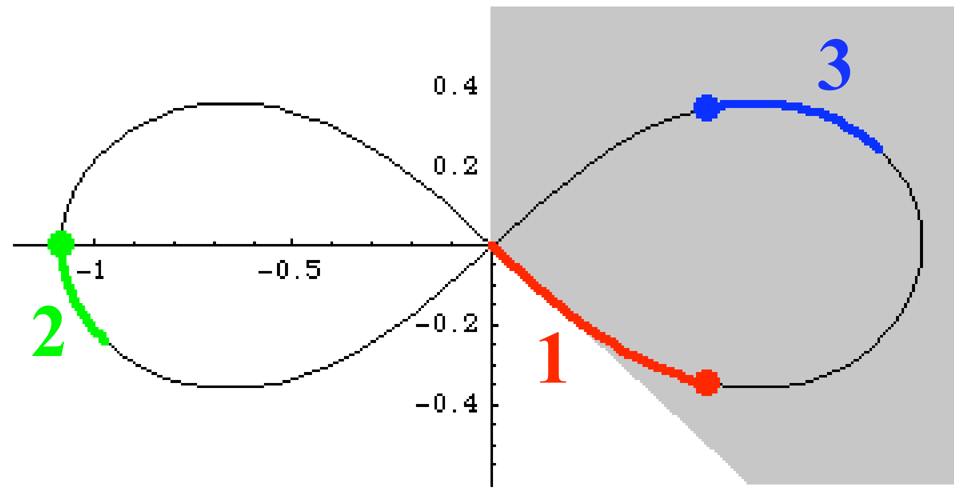
$$a_i(b \wedge c) + b_i(c \wedge a) + c_i(a \wedge b) = \begin{vmatrix} a_i & b_i & c_i \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ for } i = 1, 2.$$

$$\therefore y_1|\dot{q}_1|^3\kappa_1 = \dot{y}_1\dot{\ell}_1 - \ddot{y}_1\ell_1 > 0$$
$$\therefore \kappa_1 < 0.$$

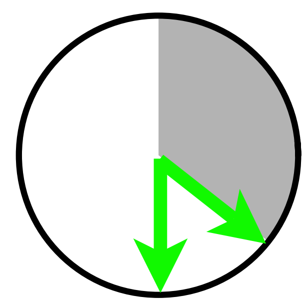
Convexity of Arc 2

Contradiction!





$$\therefore \kappa_2 > 0$$



Gauss map of $\dfrac{\dot{q}_2}{|\dot{q}_2|}$

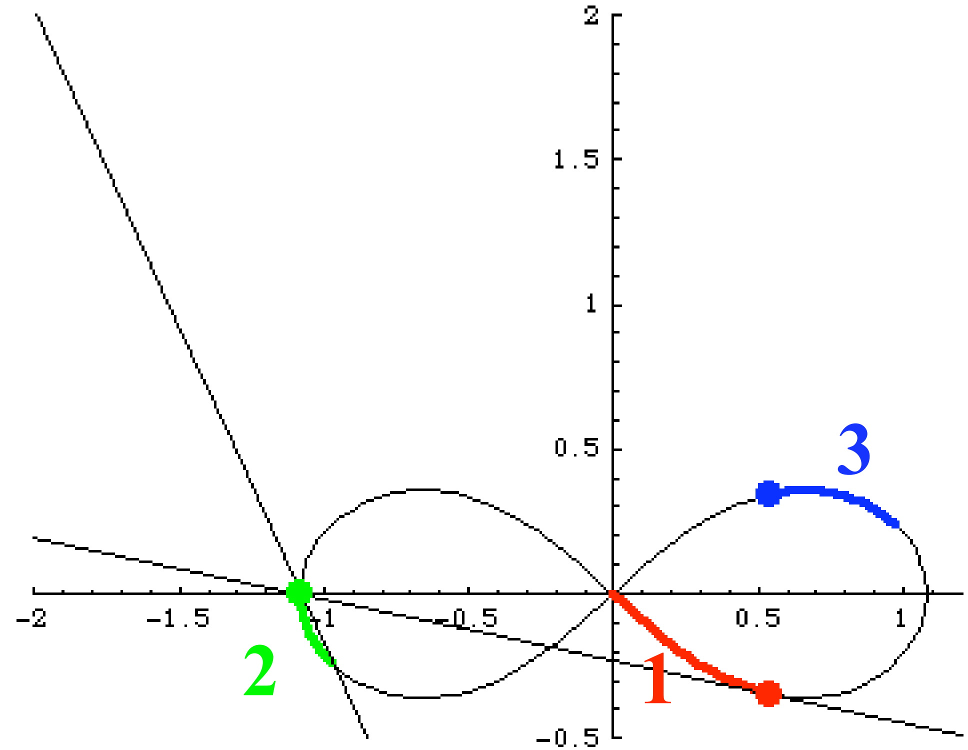
$$\ddot{x}_2 > 0 \Rightarrow \dot{x} > 0$$

Convexity of Arc 3

- Splitting lemma
- Three tangents theorem
- $\kappa_1 < 0$, $\kappa_2 > 0$
- Convexity proposition

$\Downarrow$

$\kappa_3 < 0.$



Open Questions

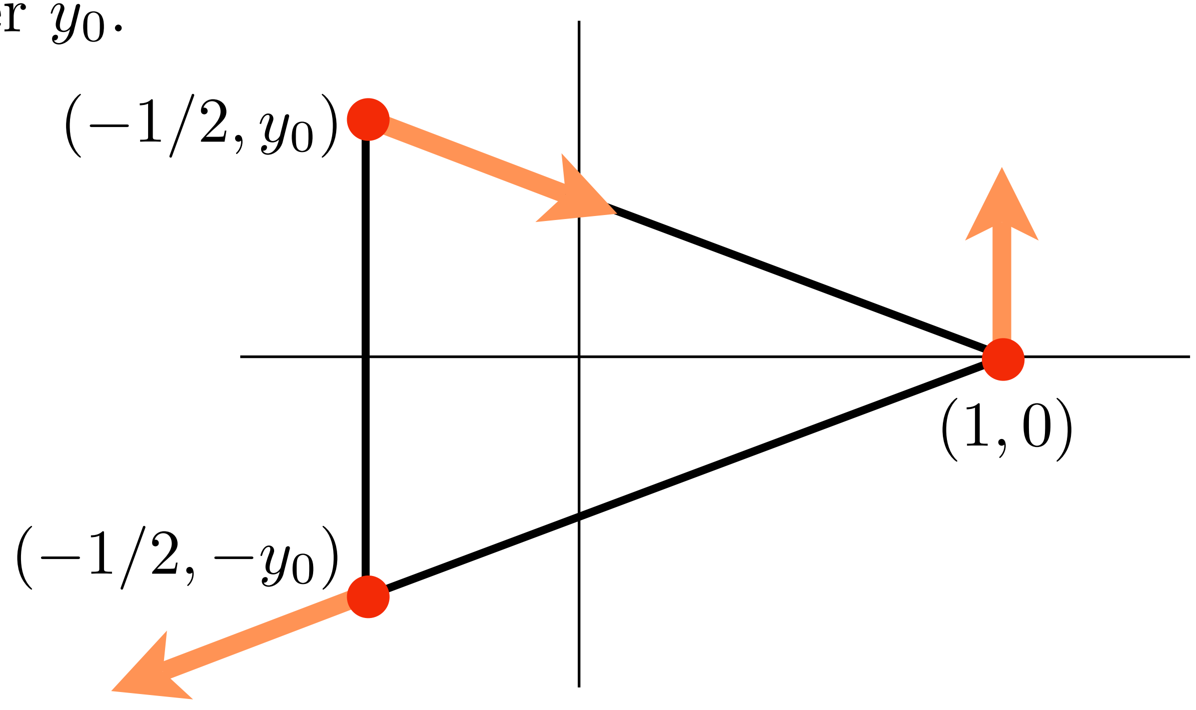
- Unicity (*The* Figure-Eight)
- Orbit $F(x,y) = 0$ polynomial?
  - Not polynomial for $-1/r$ potential (Simo)
- Choreography on Polynomial $F(x,y)=0$
  - 4: Lemniscate (FFO)
  - 6, 8, ... ?
- Exact solution under a realistic potential

Unicity of Figure Eight

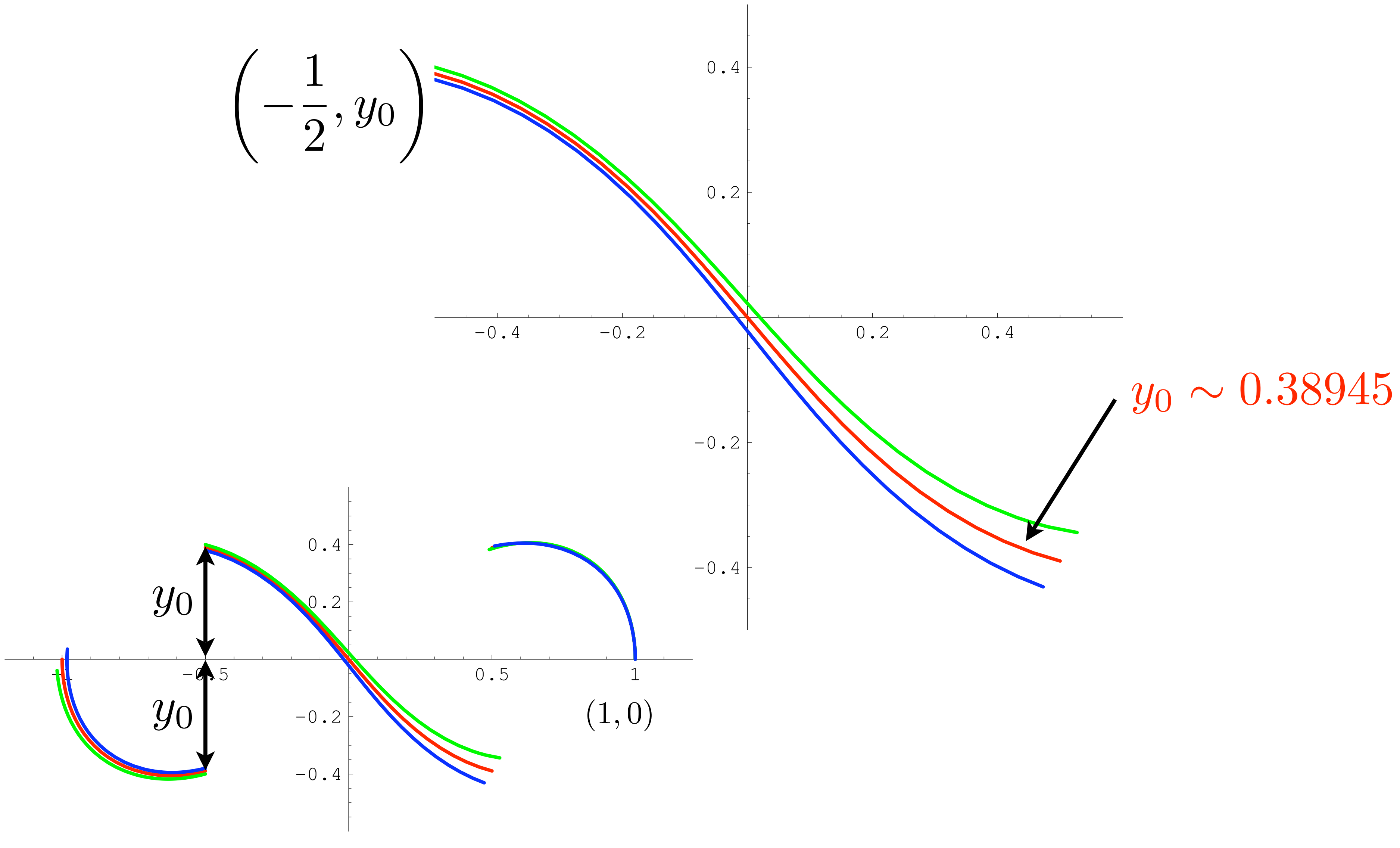
Under $V = -\dfrac{1}{2r^2}$,

$$\frac{d^2 I}{dt^2} = E \Rightarrow I = 2Et^2 + c_1 t + c_2 \Rightarrow E = 0.$$

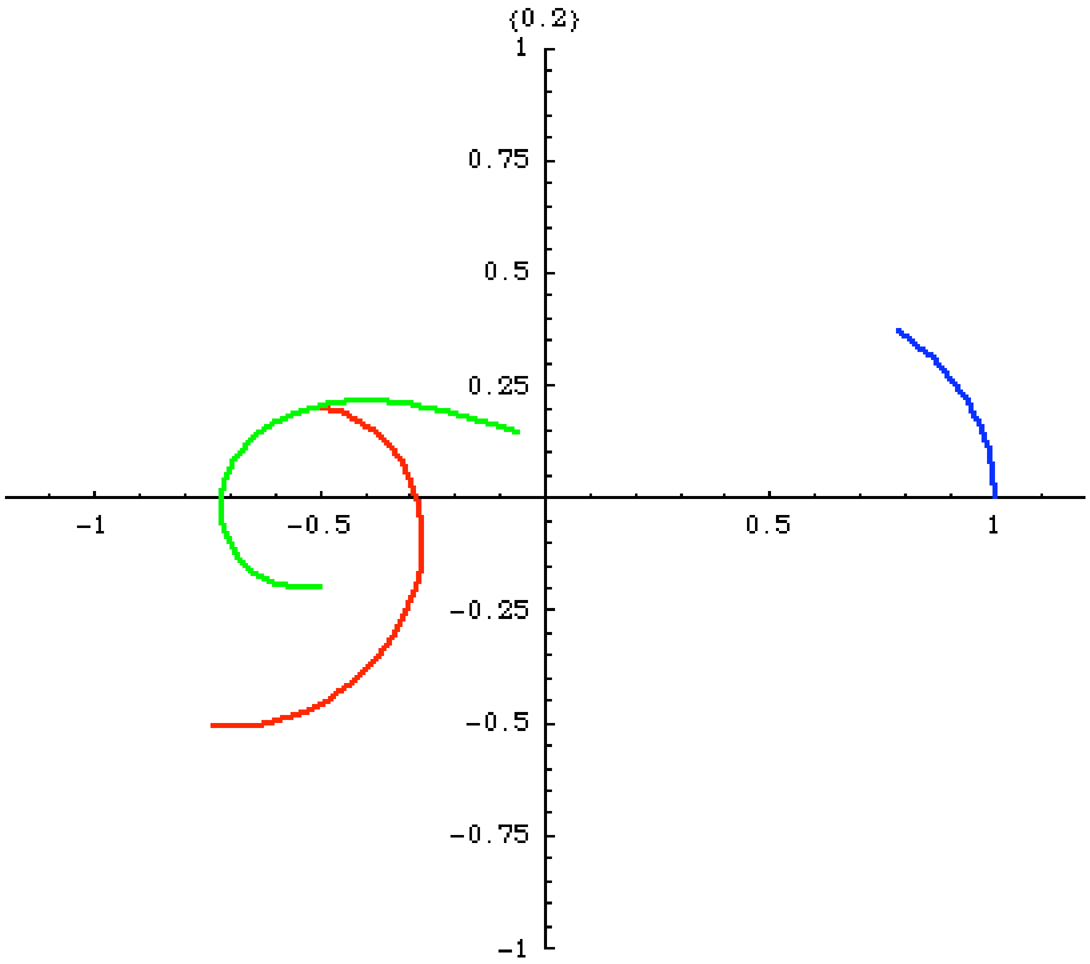
Only one parameter y_0.



$E = 0$ orbit for $V = -1/(2r^2)$



$E = 0$ orbit for $\alpha = -2$



$$0.2 \leq y_0 \leq 0.8$$

Is the Curve Algebraic?

Figure eight curve $q = (x, y)$,

$$F(x, y) = \sum_{1 \le i+j \le n} c_{ij} x^{2i} y^{2j} = 0 \quad ?$$

Simó: $2n \neq 4, 6, 8$ under $V = -1/r$, numerically.

I think, we can prove $2n \neq 4, 6, 8$.
Because ...

For $V = -1/r^2$ or $\log r$ $\quad ?$

Exact Figure-Eight Solution