The modularity of Siegel's zeta functions

Kazunari Sugiyama（杉山和成）

January 25, 2023

RIMS conference "Analytic and arithmetic aspects of automorphic representations"

History and Motivation

Riemann's zeta function (1859)

Figure 1: B. Riemann (1826-1866)

In 1859, Riemann proved that the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

initially defined in the region $\Re(s) > 1$, has a meromorphic continuation to $\mathbb{C}$, and satisfies the functional equation

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1-s).$$

Epstein's zeta functions (1903, 1907)

Figure 2: P. Epstein (1871-1939)

Let P be a positive definite symmetric matrix of degree m. Epstein defined the zeta function

$$\zeta_P(s) = \sum_{a \in \mathbb{Z}^m \setminus \{0\}} \frac{1}{P[a]^s} \quad (P[a] = {}^t a P a),$$

initially defined in the region $\Re(s) > \frac{m}{2}$. It has a memorphic continuation to $\mathbb{C}$, and satisfies the functional equation

$$\xi_{P^{-1}}\left(\frac{m}{2} - s\right) = (\det P)^{1/2} \xi_P(s),$$

where $\xi_P(s) = \pi^{-s}\Gamma(s)\zeta_P(s)$.

Hamburger's converse theorem (1921)

H. Hamburger (1889-1956) proved the following theorem *:

Let $h(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *and* $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ *be absolutely convergent for* $\Re(s) > 1$*, and suppose that both* $(s-1)h(s)$ *and* $(s-1)g(s)$ *are entire functions of finite order. Assume further that the functional equation*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)h(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)g(1-s)$$

holds. Then, in fact, $h(s) = g(s) = a_1\zeta(s)$*.*

---

*Acutually Hamburger proved a more general statement.

Hecke's converse theorem (1936)

Figure 3: E. Hecke (1887-1947)

Hecke greatly generalized Hamburger's theorem. Let $\lambda > 0, k > 0, C = \pm 1$. Denote by $M(\lambda, k, C)$ be the space of holomorphic functions $f(z)$ on the upper half plane $\mathcal{H}$ satisfying

- $f(z+\lambda) = f(z)$,
- $f(-\frac{1}{z}) = C(\frac{z}{i})^k f(z)$, and
- $f(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{\lambda}}$.

Hecke's converse theorem (1936)

For a given complex sequence $\{a_n\}_{n\geq 0}$ of polynomial growth, we set

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s).$$

Hecke proved that *the following two conditions are equivalent:*

(A) $\Phi(s) + \dfrac{a_0}{s} + \dfrac{Ca_0}{k-s}$ *is EBV and satisfies the functional equation* $\Phi(s) = C\Phi(k-s)$*.*

(B) $f(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{\lambda}} \in M(\lambda, k, C)$.

The modularity of Epstein's zeta functions

For a positive P of degree m and $z \in \mathcal{H}$, set

$$\theta_P(z) = \sum_{a \in \mathbb{Z}^m \setminus \{0\}} \exp(\pi i P[a] \cdot z).$$

We have

$$\begin{aligned}\xi_P(s) &= \int_1^\infty (\theta_P(iy) - 1) y^s \frac{dy}{y} - \frac{1}{s} \\ &+ (\det P)^{-1/2} \left\{ \int_1^\infty (\theta_{P^{-1}}(iy) - 1) y^{\frac{m}{2} - s} \frac{dy}{y} - \frac{1}{\frac{m}{2} - s} \right\},\end{aligned}$$

from which we obtain the functional equation of $\xi_P(s)$.

The modularity of Epstein's zeta functions

Assume further that P is even integral with $\det P = 1$. (In this case, m is a multiple of 8.) Then,

$$\theta_P\left(-\frac{1}{z}\right) = z^{m/2}\theta_P(z), \quad \theta_P(z+1) = \theta_P(z),$$

i.e., $\theta_P(z)$ is a modular form of $m/2$. Moreover,

$$\theta_P(z) = \sum_{l=0}^{\infty} r_P(2l)\exp(2\pi i l z),$$

where $r_P(2l)$ is the *representation number* defined by

$$r_P(2l) = \sharp\{a \in \mathbb{Z}^m\,;\, P[a] = 2l\}.$$

Siegel's zeta functions (1938, 1939)

Figure 4: C.L. Siegel (1896-1981)

Let $\boldsymbol{Y}$ be a non-degenerate half-integral symmetric matrix of degree $\boldsymbol{m}$, and put $\boldsymbol{V}_{\pm} = \{\boldsymbol{v} \in \mathbb{R}^{\boldsymbol{m}}\,;\, \mathbf{sgn}\,\boldsymbol{Y}[\boldsymbol{v}] = \pm\}$. Then Siegel's zeta functions are defined by

$$\boldsymbol{\zeta}_{\pm}(\boldsymbol{s}) = \sum_{\boldsymbol{v} \in \boldsymbol{SO}(\boldsymbol{Y})_{\mathbb{Z}} \backslash (\mathbb{Z}^{\boldsymbol{m}} \cap \boldsymbol{V}_{\pm})} \frac{\boldsymbol{\mu}(\boldsymbol{v})}{|\boldsymbol{Y}[\boldsymbol{v}]|^{\boldsymbol{s}}},$$

where $\boldsymbol{\mu}(\boldsymbol{v})$ is a certain volume of $\boldsymbol{SO}(\boldsymbol{Y})_{\boldsymbol{v},\mathbb{R}} / \boldsymbol{SO}(\boldsymbol{Y})_{\boldsymbol{v},\mathbb{Z}}$.

Siegel's comment on the modularity

Siegel proved the analytic properties such as the functional equation, and made the following remark in a 1938 paper:

> *Will man die Transformationstheorie von* $f(\mathfrak{S}, x)$ *für beliebige Modulsubstitutionen entwickeln, so hat man außer* $\zeta_1(\mathfrak{S}, s)$ *auch analog gebildete Zetafunktionen mit Restklassen-Chrakteren zu untersuchen. Die zum Beweise der Sätze 1,2,3 führenden Überlegungen lassen sich ohne wesentiche Schwierigkeit auf den allgemeinen Fall übertragen. Vermöge der Mellinschen Transformation erhält man dann das wichtige Resultat, daß die durch (53) definierte Funktion* $f(\mathfrak{S}, x)$ *eine Modulform der Dimension* $\frac{n}{2}$ *und der Stufe* $2D$ *ist; dabei wird vorausgesetzt, daß* n *ungerade und* $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ *keine ternäre Nullform ist.*

Translation of Siegel's comment

A translation (with DeepL) is as follows:

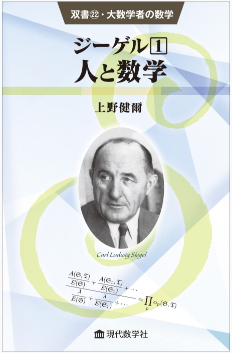
> *If one wants to develop the transformation theory of $f(\mathfrak{S}, x)$ for arbitrary modular substitutions, then in addition to $\zeta_1(\mathfrak{S}, s)$ one also has to investigate zeta functions formed analogously with residual class characters. The considerations leading to the proof of Theorems 1, 2, 3 can be transferred to the general case without any major difficulty. By virtue of the (inverse) Mellin transformation, one then obtains an important result that the function $f(\mathfrak{S}, x)$ defined by (53) is a modular form of weight $\frac{n}{2}$ and level $2D$, provided that n is odd and $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ is not a ternary zero form.*

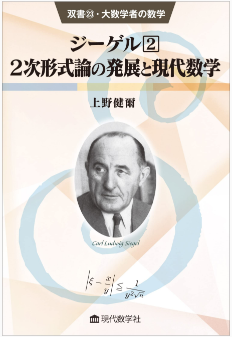
The aim of this talk

The aim of this talk is

- **to accompolish Siegel's original plan by using a Weil-type converse theorem for Maass forms, which has appeared recently.**
- **to show that "half" of Siegel's zeta functions correspond to holomorphic modular forms.**

ジーゲルについての本





Recently, Prof. Kenji Ueno (上野健爾) wrote two books (in japanese) on Siegel, both on his life and his mathematics.

Maass' converse theorem (1949)

Figure 5: H. Maaß(1911-1992)

Maaß introduced the notion of non-holomorphic modular forms (*Maass forms*). He proved a converse theorem for Maass forms, and applied it to Siegel's zeta functions. It was shown that in a very special case (when $\boldsymbol{Y}$ is diagonal of even degree with $\det \boldsymbol{Y} = \boldsymbol{1}$), Siegel's zeta functions can be expressed as the product of two standard Dirichlet series such as $\boldsymbol{\zeta(s)}$ and $\boldsymbol{L(s,\chi)}$.

Siegelsche HauptSatz **(1951)**

In terms of Siegel's zeta functions $\zeta_\pm(s)$, Siegel's main theorem states that the coefficients

$$\boldsymbol{M(Y;\pm n) = \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash (\mathbb{Z}^m \cap V_\pm) \\ Y[v] = \pm n}} \mu(v) \qquad (n = 1, 2, 3, \dots)}$$

can be expressed as the product of local representation densities over all primes. Siegel called $M(Y;n)$ the measures of representations (*Darstellungsmaß*). In the course of the proof of the main theorem, it is shown that the measures $M(Y;n)$ appear as Fourier coefficients of some real analytic automorphic form, which is an integral of indefinite theta series.

Ibukiyama's explicit formula

In his book 『保型形式特論』 (*Topics on modular forms*), T. Ibukiyama proved that when m is even, Siegel's zeta functions can be expressed as a $\mathbb{Q}$-linear combinations of

$$a^s L(s,\chi_1) L\left(s - \frac{m}{2} + 1, \chi_2\right),$$

where a is a positive rational number, and χ_1, χ_2 are real Dirichlet characters.
In the proof, Siegel's main theorem is used.

A Weil-type converse theorem for Maass forms

Weil's converse theorem (1967)

Figure 6: A. Weil (1906-1998)

Weil characterized modular forms for the congruence subgroup $\Gamma_0(N)$ by *twisting* the series $\phi(s)$ by Dirichlet characters. In Weil's converse theorem, for each *primitive* character ψ, the analytic properties of

$$\Lambda(s,\psi) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}\frac{\psi(n)a_n}{n^s}$$

are assumed.

Modular forms of half integral weight (1973)

Figure 7:
G. Shimura
(1930-2019)

In a 1973 paper, which appeared in Annals of Math., Shimura studied modular forms of half integral weight. In the last section of the paper, he mentioned that as in Weil's paper, one can characterize modular forms of half-integral weight by analytic properties of twisted L-functions. Shimura wrote *"we do not give here an explicit statement, which is rather obvious"*.

The purpose of this section

In this section, we define Maass forms of integral and half-integral weight, and recall a Weil-type converse theorem that is proved in

> *T. Miyazaki, F. Sato, T. Ueno and S., Converse theorems for automorphic distributions and Maass forms of level* N*, Res. number theory 6:6 (2020).*

Definition of Maass forms

Let $\Gamma = SL_2(\mathbb{Z})$ be the modular group, and for a positive integer N, we denote by $\Gamma_0(N)$ the congruence subgroup. As usual, Γ acts on $\mathcal{H}$ by the linear fractional transformation. We put $j(\gamma, z) = cz + d$, and define $\theta(z)$ and $J(\gamma, z)$ by

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z), \qquad J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}.$$

Then it is well-known that

$$J(\gamma, z) = \varepsilon_d^{-1} \cdot \left(\frac{c}{d}\right) \cdot (cz+d)^{1/2} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where

$$\varepsilon_d = \begin{cases} 1 & (d \equiv 1 \pmod 4), \\ i & (d \equiv 3 \pmod 4). \end{cases}$$

Definition of Maass forms

For an integer ℓ, the hyperbolic Laplacian $\Delta_{\ell/2}$ of weight $\ell/2$ on $\mathcal{H} = \{z = x + iy \in \mathbb{C}\, ;\, y > 0\}$ is defined by

$$\Delta_{\ell/2} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{i\ell y}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Let χ be a Dirichlet character mod N. Then we use the same symbol χ to denote the character of $\Gamma_0(N)$ defined by

$$\chi(\gamma) = \chi(d) \qquad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Definition of Maass forms

Definition 1

Let $\ell \in \mathbb{Z}$, and N be a positive integer, with $4|N$ when ℓ is odd. A complex-valued C^∞-function $F(z)$ on $\mathcal{H}$ is called a Maass form for $\Gamma_0(N)$ of weight $\ell/2$ with character χ, if the following three conditions are satisfied;

1. *for every $\gamma \in \Gamma_0(N)$,*

$$F(\gamma z) = \begin{cases} \chi(\gamma) j(\gamma, z)^{\ell/2} \cdot F(z) & (\ell \text{ is even}) \\ \chi(\gamma) J(\gamma, z)^{\ell} \cdot F(z) & (\ell \text{ is odd}) \end{cases},$$

2. *$\Delta_{\ell/2} F = \Lambda \cdot F$ with some $\Lambda \in \mathbb{C}$,*
3. *F is of moderate growth at every cusp.*

A Weil-type converse theorem

Let λ be a complex number with $\lambda \notin 1 - \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\alpha = \{\alpha(n)\}_{n \in \mathbb{Z} \setminus \{0\}}$ and $\beta = \{\beta(n)\}_{n \in \mathbb{Z} \setminus \{0\}}$ be complex sequences of polynomial growth. For α, β, we can define the L-functions $\xi_\pm(\alpha; s), \xi_\pm(\beta; s)$ and the completed L-functions $\Xi_\pm(\alpha; s), \Xi_\pm(\beta; s)$ by

$$\xi_\pm(\alpha; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{n^s}, \quad \Xi_\pm(\alpha; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha; s),$$

$$\xi_\pm(\beta; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)}{n^s}, \quad \Xi_\pm(\beta; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta; s).$$

A Weil-type converse theorem

Now we assume the following conditions [A1] – [A4]:

[A1] $\xi_\pm(\alpha; s), \xi_\pm(\beta; s)$ have meromorphic continuations to the whole s-plane, and $(s-1)(s-2+2\lambda)\xi_\pm(\alpha; s)$ and $(s-1)(s-2+2\lambda)\xi_\pm(\beta; s)$ are entire functions, which are of finite order in any vertical strip.

[A2] The residues of $\xi_\pm(\alpha; s)$ and $\xi_\pm(\beta; s)$ at $s = 1$ satisfy

$$\operatorname*{Res}_{s=1} \xi_+(\alpha; s) = \operatorname*{Res}_{s=1} \xi_-(\alpha; s),$$
$$\operatorname*{Res}_{s=1} \xi_+(\beta; s) = \operatorname*{Res}_{s=1} \xi_-(\beta; s).$$

A Weil-type converse theorem

[A3] The following functional equation holds:

$$\gamma(s)\begin{pmatrix}\Xi_+(\alpha;s)\\ \Xi_-(\alpha;s)\end{pmatrix}$$
$$= N^{2-2\lambda-s}\cdot\Sigma(\ell)\cdot\gamma(2-2\lambda-s)\begin{pmatrix}\Xi_+(\beta;2-2\lambda-s)\\ \Xi_-(\beta;2-2\lambda-s)\end{pmatrix},$$

where $\gamma(s)$ and $\Sigma(\ell)$ are defined by

$$\gamma(s)=\begin{pmatrix}e^{\pi si/2} & e^{-\pi si/2}\\ e^{-\pi si/2} & e^{\pi si/2}\end{pmatrix},\qquad \Sigma(\ell)=\begin{pmatrix}0 & i^{\ell}\\ 1 & 0\end{pmatrix}.$$

A Weil-type converse theorem

[A4] If $\lambda = \frac{q}{2}$ $(q \in \mathbb{Z}_{\geq 0},\ q \geq 4)$, then

$$\xi_+(\alpha; -k) + (-1)^k \xi_-(\alpha; -k) = 0 \quad (k = 1, 2, \ldots, q-3).$$

Under the assumptions [A1] – [A4], we define $\alpha(0)$, $\beta(0)$, $\alpha(\infty)$, $\beta(\infty)$ by

$$\begin{aligned} \alpha(0) &= -\xi_e(\alpha; 0) \\ \alpha(\infty) &= \frac{N}{2} \operatorname*{Res}_{s=1} \xi_e(\beta; s), \\ \beta(0) &= -\xi_e(\beta; 0) \\ \beta(\infty) &= \frac{i^{-\ell}}{2} \operatorname*{Res}_{s=1} \xi_e(\alpha; s), \end{aligned}$$

where $\xi_e(*; s) = \xi_+(*; s) + \xi_-(*; s)$. $(* = \alpha$ or β.)

A Weil-type converse theorem

For an odd prime number r with $(N, r) = 1$ and a Dirichlet character ψ mod r, the twisted L-functions $\xi_{\pm}(\alpha, \psi; s), \Xi_{\pm}(\alpha, \psi; s)$ are defined by

$$\xi_{\pm}(\alpha, \psi; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)\tau_{\psi}(\pm n)}{n^s},$$

$$\Xi_{\pm}(\alpha, \psi; s) = (2\pi)^{-s}\Gamma(s)\xi_{\pm}(\alpha, \psi; s),$$

where $\tau_{\psi}(n)$ is the Gauss sum defined by

$$\tau_{\psi}(n) = \sum_{\substack{m \bmod r \\ (m,r)=1}} \psi(m) e^{2\pi i m n / r}.$$

$\xi_{\pm}(\beta, \psi; s), \Xi_{\pm}(\beta, \psi; s)$ are defined similarly.

A Weil-type converse theorem

Let $\mathbb{P}_N$ be a set of odd prime numbers not dividing N. For an $r \in \mathbb{P}_N$, denote by X_r the set of all Dirichlet characters mod r (including the principal character $\psi_{r,0}$). For $\psi \in X_r$, we define the Dirichlet character ψ^* by

$$\psi^*(k) = \overline{\psi(k)} \left(\frac{k}{r}\right)^{\ell}.$$

We put

$$C_{\ell,r} = \begin{cases} 1 & (\ell \text{ is even}), \\ \varepsilon_r^{\ell} & (\ell \text{ is odd}). \end{cases}$$

A Weil-type converse theorem

In the following, we fix a Dirichlet character χ mod N that satisfies $\chi(-1) = i^{\ell}$ (resp. $\chi(-1) = 1$) when ℓ is even (resp. odd).

For an $r \in \mathbb{P}_N$ and a $\psi \in X_r$, we assume some conditions on $\xi_{\pm}(\alpha, \psi; s)$ and $\xi_{\pm}(\beta, \psi^*; s)$ such as

$$\begin{aligned}
&\gamma(s) \begin{pmatrix} \Xi_+(\alpha, \psi; s) \\ \Xi_-(\alpha, \psi; s) \end{pmatrix} \\
&= \chi(r) \cdot C_{\ell,r} \cdot \psi^*(-N) \cdot r^{2\lambda-2} \cdot (Nr^2)^{2-2\lambda-s} \\
&\qquad \cdot \Sigma(\ell) \cdot \gamma(2-2\lambda-s) \begin{pmatrix} \Xi_+ (\beta, \psi^*; 2-2\lambda-s) \\ \Xi_- (\beta, \psi^*; 2-2\lambda-s) \end{pmatrix}.
\end{aligned}$$

A Weil-type converse theorem

Define the function $F_\alpha(z)$ on $\mathcal{H}$ by

$$
\begin{aligned}
F_\alpha(z) = {} & \alpha(\infty) \cdot y^{\lambda-\ell/4} \\
& + \alpha(0) \cdot i^{-\ell/2} \cdot \frac{(2\pi)2^{1-2\lambda}\Gamma(2\lambda-1)}{\Gamma\left(\lambda+\frac{\ell}{4}\right)\Gamma\left(\lambda-\frac{\ell}{4}\right)} \cdot y^{1-\lambda-\ell/4} \\
& + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha(n) \cdot \frac{i^{-\ell/2}\cdot\pi^{\lambda}\cdot|n|^{\lambda-1}}{\Gamma\left(\lambda+\frac{\operatorname{sgn}(n)\ell}{4}\right)} \cdot \mathcal{W}_{\ell,n,\lambda}(y)\cdot \mathrm{e}[nx],
\end{aligned}
$$

where $\mathcal{W}_{\ell,n,\lambda}(y) = y^{-\ell/4}\, W_{\frac{\operatorname{sgn}(n)\ell}{4},\lambda-\frac{1}{2}}\,(4\pi|n|y)$. We define $G_\beta(z)$ from β similarly.

A Weil-type converse theorem

Lemma 1 (Converse Theorem)

Then $F_\alpha(z)$ *(resp.* $G_\beta(z)$*) gives a Maass form for* $\Gamma_0(N)$ *of weight* $\frac{\ell}{2}$ *with character* χ *(resp.* $\chi_{N,\ell}$*), and eigenvalue* $(\lambda - \ell/4)(1 - \lambda - \ell/4)$, *where*

$$\chi_{N,\ell}(d) = \overline{\chi(d)} \left(\frac{N}{d}\right)^{\ell}.$$

Moreover, we have

$$F_\alpha\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-\ell/2} = G_\beta(z).$$

Analytic properties of Siegel's zeta functions

Prehomogeneous vector spaces

Siegel's calculation can be well understood in the framework of the thery of prehomogeneous vector spaces, which is developed by M. Sato and Shintani.

Figure 8: Mikio Sato (佐藤幹夫, 1928-2023) and Takuro Shintani (新谷卓郎, 1943-1980)

Prehomogeneous vector spaces

We assume that $m \geq 5$. Let Y be a non-degenerate half-integral symmetric matrix of degree m, and let p be the number of positive eigenvalues of Y. Put $SO(Y) = \{g \in SL_m(\mathbb{C}) \mid {}^t g Y g = Y\}$. We define the representation ρ of $G = GL_1(\mathbb{C}) \times SO(Y)$ on $V = \mathbb{C}^m$ by

$$\rho(\tilde{g})v = \rho(t, g)v = tgv \qquad (\tilde{g} = (t, g) \in G, v \in V).$$

Let $P(v)$ be the quadratic form on V defined by $P(v) = Y[v] = {}^t v Y v$. Then $V - S$ is a single $\rho(G)$-orbit, where $S = \{v \in V \mid P(v) = 0\}$. That is, (G, ρ, V) is a (regular) prehomogeneous vector space.

Prehomogeneous vector spaces

We identify the dual space V^* of V with V itself via the inner product $\langle v, v^* \rangle = {}^t v v^*$. Then the dual triplt (G, ρ^*, V^*) is given by

$$\rho^*(\tilde{g})v^* = \rho^*(t, g)v^* = t^{-1} \cdot {}^t g^{-1} v^*.$$

We define the quadratic form $P^*(v^*)$ on V^* by $P^*(v^*) = \frac{1}{4} Y^{-1}[v^*] = \frac{1}{4} \cdot {}^t v^* \, Y^{-1} v^*$. Then, $V - S^*$ is a single $\rho^*(G)$-oribit, where S^* is the zero set of P^*.

Local zeta functions

For $\epsilon, \eta = \pm$, we put

$$V_\epsilon = \{v \in V_{\mathbb{R}} \mid \operatorname{sgn} P(v) = \epsilon\},$$
$$V^*_\eta = \{v^* \in V_{\mathbb{R}} \mid \operatorname{sgn} P^*(v^*) = \eta\}.$$

Denote by $\mathcal{S}(V_{\mathbb{R}})$ the space of rapidly decreasing functions on $V_{\mathbb{R}}$. For $f, f^* \in \mathcal{S}(V_{\mathbb{R}})$ and $\epsilon, \eta = \pm$, we put

$$\Phi_\epsilon(f; s) = \int_{V_\epsilon} f(v)|P(v)|^{s-\frac{m}{2}} dv,$$
$$\Phi^*_\eta(f^*; s) = \int_{V^*_\eta} f^*(v^*)|P^*(v^*)|^{s-\frac{m}{2}} dv^*.$$

We define the Fourier transform $\widehat{f}(v^*)$ of $f \in \mathcal{S}(V_{\mathbb{R}})$ by

$$\widehat{f}(v^*) = \int_{V_{\mathbb{R}}} f(v)\mathrm{e}[\langle v, v^* \rangle] dv.$$

Local Functional Equation

Lemma 2

Let p be the number of positive eigenvalues of Y, and put $D = \det(2Y)$. Then we have

$$\begin{pmatrix} \Phi_+^*(\widehat{f}\,; s) \\ \Phi_-^*(\widehat{f}\,; s) \end{pmatrix}$$

$$= \Gamma\left(s + 1 - \frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}} \cdot 2^{-2s+\frac{m}{2}} \cdot \pi^{-2s+\frac{m}{2}-1}$$

$$\times \begin{pmatrix} \sin\pi\left(\frac{p}{2} - s\right) & \sin\frac{\pi p}{2} \\ \sin\frac{\pi(m-p)}{2} & \sin\pi\left(\frac{m-p}{2} - s\right) \end{pmatrix} \begin{pmatrix} \Phi_+\left(f; \frac{m}{2} - s\right) \\ \Phi_-\left(f; \frac{m}{2} - s\right) \end{pmatrix}.$$

Normalizations of measures

Let dx (resp. $d\lambda$) be the measure on $GL_m(\mathbb{R})$ (resp. $\mathrm{Sym}_m(\mathbb{R})$) defined by

$$dx = |\det x|^{-m} \prod_{1\le i,j\le m} dx_{ij},$$

$$d\lambda = |\det \lambda|^{-\frac{m+1}{2}} \prod_{1\le i\le j\le m} d\lambda_{ij}.$$

We normalize a Haar measure d^1g on the Lie group $SO(Y)_{\mathbb{R}}$ in such a way that for all $F(x) \in L^1(GL_m(\mathbb{R}))$,

$$\int_{GL_m(\mathbb{R})} F(x)dx$$

$$= \int_{SO(Y)_{\mathbb{R}} \backslash GL_m(\mathbb{R})} d\lambda({}^t\dot{x}Y\dot{x}) \int_{SO(Y)_{\mathbb{R}}} F(g\dot{x})d^1g.$$

Normalizations of measures

Let
$$SO(Y)_v = \{g \in SO(Y) \mid gv = v\}$$
be the isotropy subgroup at $v \in V - S$, which is reductive. For $v \in V_\epsilon$, there exists a Haar measure $d\mu_v$ on $SO(Y)_{v,\mathbb{R}}$ such for all $H(t,g) \in L^1(G_\mathbb{R})$,

$$\begin{aligned}
&\int_0^\infty d^\times t \int_{SO(Y)_\mathbb{R}} H(t,g) d^1 g \\
&= \int_0^\infty \int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{R}}} |P(\rho(t,\dot{g})v)|^{-\frac{m}{2}} d(\rho(t,\dot{g})v) \\
&\qquad \times \int_{SO(Y)_{v,\mathbb{R}}} H(t,\dot{g}h) d\mu_v(h).
\end{aligned}$$

Definition of the density $\mu(v)$

For $v \in V_{\mathbb{Q}} - S_{\mathbb{Q}}$, we put

$$\mu(v) = \int_{SO(Y)_{v,\mathbb{R}}/SO_{v,\mathbb{Z}}} d\mu_v(h).$$

Since it is assumed that $m \geq 5$, the generic isotropy subgroup $SO(Y)_v$ is a semisimple algebraic group, and thus we have $\mu(v) < +\infty$ by a theorem of Borel and Harish-Chandra.

Schwartz-Bruhat functions on $V_{\mathbb{Q}}$

We call a function $\phi : V_{\mathbb{Q}} \to \mathbb{C}$ a *Schwartz-Bruhat* function if

1. there exists a positive integer M such that $\phi(v) = 0$ for $v \notin \frac{1}{M} V_{\mathbb{Z}}$, and
2. there exists a positive integer N such that if $v, w \in V_{\mathbb{Q}}$ satisfy $v - w \in N V_{\mathbb{Z}}$. then $\phi(v) = \phi(w)$.

The totallity of Schwartz-functions on $V_{\mathbb{Q}}$ is denoted by $\mathcal{S}(V_{\mathbb{Q}})$.

Poisson summation formula

We define the Fourier transform $\widehat{\phi} \in \mathcal{S}(V_{\mathbb{Q}})$ by

$$\widehat{\phi}(v^*) = \frac{1}{[V_{\mathbb{Z}} : rV_{\mathbb{Z}}]} \sum_{v \in V_{\mathbb{Q}}/rV_{\mathbb{Z}}} \phi(v)\mathrm{e}[-\langle v, v^* \rangle],$$

where r is a sufficiently large positive integer such that the value $\phi(v)\mathrm{e}[-\langle v, v^* \rangle]$ depends only on the residue class $v \bmod rV_{\mathbb{Z}}$.

Lemma 3 (Poisson summation formula)

For $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ and $f \in \mathcal{S}(V_{\mathbb{R}})$,

$$\sum_{v^* \in V_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(v^*) = \sum_{v \in V_{\mathbb{Q}}} \phi(v) f(v).$$

Siegel's zeta functions

Definition 4 (Siegel's zeta functions)

Let $\epsilon, \eta = \pm$. For $\phi, \phi^* \in \mathcal{S}(V_{\mathbb{Q}})$, we define *Siegel's zeta functions* $\zeta_\epsilon(\phi; s)$ and $\zeta_\eta^*(\phi^*; s)$ by

$$\zeta_\epsilon(\phi; s) = \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\epsilon \cap V_{\mathbb{Q}}} \frac{\phi(v)\mu(v)}{|P(v)|^s},$$

$$\zeta_\eta^*(\phi^*; s) = \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V_\eta^* \cap V_{\mathbb{Q}}} \frac{\phi^*(v^*)\mu^*(v^*)}{|P^*(v^*)|^s}.$$

These zeta functions converge absolutely for $\Re(s) > m/2$.

Zeta integrals

For $\phi, \phi^* \in \mathcal{S}(V_{\mathbb{Q}})$ and $f, f^* \in \mathcal{S}(V_{\mathbb{R}})$, we define the zeta integrals by

$$\begin{aligned}
&Z(f,\phi;s)\\
&= \int_0^\infty t^{2s} d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v \in V_{\mathbb{Q}} - S_{\mathbb{Q}}} \phi(v) f(\rho(t,g)v) d^1 g,\\
&Z^*(f^*,\phi;s)\\
&= \int_0^\infty t^{-2s} d^\times t\\
&\qquad \times \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v^* \in V_{\mathbb{Q}} - S^*_{\mathbb{Q}}} \phi^*(v^*) f^*(\rho^*(t,g)v^*) d^1 g.
\end{aligned}$$

Integral representatons of Siegel's zeta functions

Lemma 5 (Integral representations of the zeta functions)

Assume that $\phi, \phi^* \in \mathcal{S}(V_{\mathbb{Q}})$ *are* $SO(Y)_{\mathbb{Z}}$*-invariant. For* $\Re(s) > \frac{m}{2}$*, we have*

$$Z(f, \phi; s) = \sum_{\epsilon=\pm} \zeta_\epsilon(\phi; s)\Phi_\epsilon(f; s),$$

$$Z^*(f^*, \phi^*; s) = \sum_{\eta=\pm} \zeta^*_\eta(\phi^*; s)\Phi^*_\eta(f^*; s).$$

Analytic continuations of Siegel's zeta functions

In the following, we assume that $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ is $SO(Y)_{\mathbb{Z}}$-invariant.

Theorem 6

The zeta functions $\zeta_{\epsilon}(\phi; s)$ and $\zeta^*_{\eta}(\widehat{\phi}; s)$ have analytic continuations of s in $\mathbb{C}$, and the zeta functions multiplied by $(s-1)(s-\frac{m}{2})$ are entire functions of s of finite order in any vertical strip.

The functional equation of Siegel's zeta functions

Theorem 7

The zeta functions $\zeta_\epsilon(\phi; s)$ and $\zeta^*_\eta(\widehat{\phi}; s)$ satisfy the following functional equation:

$$
\begin{pmatrix} \zeta_+\left(\phi; \frac{m}{2} - s\right) \\ \zeta_-\left(\phi; \frac{m}{2} - s\right) \end{pmatrix}
= \Gamma\left(s + 1 - \frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}} \cdot 2^{-2s+\frac{m}{2}} \cdot \pi^{-2s+\frac{m}{2}-1}
\times \begin{pmatrix} \sin\pi\left(\frac{p}{2} - s\right) & \sin\frac{\pi(m-p)}{2} \\ \sin\frac{\pi p}{2} & \sin\pi\left(\frac{m-p}{2} - s\right) \end{pmatrix}
\begin{pmatrix} \zeta^*_+(\widehat{\phi}; s) \\ \zeta^*_-(\widehat{\phi}; s) \end{pmatrix}.
$$

Residues

Lemma 8

We have

$$\operatorname*{Res}_{s=\frac{m}{2}} \zeta_\epsilon(\phi; s) = \widehat{\phi}(0) \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g,$$

$$\operatorname*{Res}_{s=\frac{m}{2}} \zeta_\eta^*(\widehat{\phi}; s) = \phi(0) \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g.$$

We also have some formulas for $\operatorname*{Res}_{s=1} \zeta_\epsilon(\phi; s)$ *and* $\operatorname*{Res}_{s=1} \zeta_\eta^*(\widehat{\phi}; s)$.

Main results

Level

Let $D = \det(2Y)$ and N be the level of $2Y$. That is, N is the smallest positive integer such that $N(2Y)^{-1}$ is even integral. We define a half-integral symmetric matrix $\widehat{Y}$ by

$$\widehat{Y} = \frac{1}{4} N Y^{-1}.$$

We define the quadratic form $P(v)$ on V by $P(v) = Y[v] = {}^t v Y v$, and the quadratic form $\widehat{P}(v^*)$ on V^* by

$$\widehat{P}(v^*) = \widehat{Y}[v^*].$$

Character

We define a field $\boldsymbol{K}$ by

$$\boldsymbol{K} = \begin{cases} \mathbb{Q}(\sqrt{(-1)^{m/2}D}) & (m \equiv 0 \pmod 2) \\ \mathbb{Q}(\sqrt{2|D|}) & (m \equiv 1 \pmod 2) \end{cases},$$

and χ_K be the Kronecker symbol associated to $\boldsymbol{K}$.

Specializing ϕ

For an odd prime r with $(r, N) = 1$ and a Dirichlet character ψ of modulus r, we define the function $\phi_{\psi,P}(v)$ on $V_{\mathbb{Q}}$ by

$$\phi_{\psi,P}(v) = \tau_\psi(P(v)) \cdot \mathrm{ch}_{\mathbb{Z}^m}(v),$$

where $\tau_\psi(P(v))$ is the Gauss sum. We have

$$\zeta_\varepsilon(\phi_{\psi,P}(v); s) = \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\varepsilon \cap V_{\mathbb{Z}}} \frac{\tau_\psi(P(v))\mu(v)}{|P(v)|^s}.$$

Fourier transform of $\phi_{\psi,P}$

Lemma 9 (Stark)

Let $\widehat{\phi_{\psi,P}}(v^*)$ be the Fourier transform of $\phi_{\psi,P}$. Then the support of $\widehat{\phi_{\psi,P}}(v^*)$ is contained in $r^{-1}\mathbb{Z}^m$, and for $v^* \in \mathbb{Z}^m$, we have

$$\begin{aligned}&\widehat{\phi_{\psi,P}}(r^{-1}v^*)\\ &\quad = r^{-m/2}\chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \tau_{\psi^*}(\widehat{P}(v^*)),\end{aligned}$$

where $\psi^*(k) = \overline{\psi(k)} \left(\frac{k}{r}\right)^m$ and

$$C_{2p-m,r} = \begin{cases} 1 & (m \equiv 0 \pmod 2) \\ \varepsilon_r^{2p-m} & (m \equiv 1 \pmod 2) \end{cases}.$$

Measures of representations (*Darstellungsmaß*)

Definition 10 (Siegel)

For $n \in \mathbb{Z} \setminus \{0\}$, we put

$$M(P;n) = \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm} \cap V_{\mathbb{Z}} \\ P(v)=n}} \mu(v),$$

$$M^*(\widehat{P};n) = \sum_{\substack{v^* \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm}^* \cap V_{\mathbb{Z}} \\ \widehat{P}(v^*)=n}} \mu^*(v^*).$$

We call $M(P;n)$ (resp. $M^*(\widehat{P};n)$) the measures of representation (*Darstellungsmaß*) of n by P (resp. $\widehat{P}$).

The sums in the definition are finite sums by a theorem of Borel and Harish-Chandra.

Volumes $\sigma(a_i)$ on the singular set

Let $S_{1,\mathbb{R}} = \{v \in V_\mathbb{R} \mid P(v) = 0, v \neq 0\}$. For $v \in S_{1,\mathbb{R}}$, we can define a volume $\sigma(v)$ of $SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}$ in a certain way. In general, $SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Z}}$ is not a finite set, while

$$\{v \in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Z}}\,;\, v \text{ is primitive}\}$$

is a finite set. Let $a_1, \ldots, a_h$ be a complete system of representatives of this set, and we get volumes $\sigma(a_i)\,(i = 1, \cdots, h)$.

Main Theorem

Assume that at least one of m or p is an odd integer. Take an integer ℓ with $\ell \equiv 2p - m \pmod 4$. Define C^∞-functions $F(z)$ on $\mathcal{H}$ by

$$\begin{aligned}
F(z) &= y^{(m-\ell)/4} \cdot \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g \\
&+ (-1)^{(2p-m-\ell)/4} \zeta(m-2) \\
&\quad \cdot \sum_{i=1}^{h} \frac{\sigma(a_i)}{|D|^{\frac{1}{2}}} \times \frac{(2\pi) 2^{1-\frac{m}{2}} \Gamma(\frac{m}{2}-1)}{\Gamma\left(\frac{m+\ell}{4}\right) \Gamma\left(\frac{m-\ell}{4}\right)} \cdot y^{1-(m+\ell)/4} \\
&+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^{(2p-m-\ell)/4} \cdot \frac{M(P;n)}{|D|^{\frac{1}{2}}} \frac{\pi^{\frac{m}{4}} \cdot |n|^{-\frac{m}{4}}}{\Gamma\left(\frac{m+\operatorname{sgn}(n)\ell}{4}\right)} \\
&\quad \times y^{-\frac{\ell}{4}} W_{\frac{\operatorname{sgn}(n)\ell}{4}, \frac{m}{4}-\frac{1}{2}} (4\pi |n| y) \mathrm{e}[nx].
\end{aligned}$$

Main Theorem

Theorem 11

$F(z)$ is a Maass form for $\Gamma_0(N)$ of weight $\ell/2$ with eigenvalue $(m-\ell)(4-m-\ell)/16$ and character χ_K. We have a similar result for $G(z)$ that can be constructed from $M^*(\widehat{P};n)$, and we have

$$F\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-\ell/2} = G(z).$$

Lower (Upper) triangular case

Assume that the number of negative eigenvalues of $\boldsymbol{Y}$ is even; that is, $\boldsymbol{m} - \boldsymbol{p}$ is an even integer. Then the first row of the functional equation is of the following form:

$$\begin{aligned}
&\zeta_+\left(\phi; \frac{m}{2} - s\right) \\
&= \Gamma\left(s + 1 - \frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}} \cdot 2^{-2s+\frac{m}{2}} \cdot \pi^{-2s+\frac{m}{2}-1} \\
&\qquad \times \sin\pi\left(\frac{p}{2} - s\right)\zeta_+^*(\widehat{\phi}; s).
\end{aligned}$$

This suggests that $\zeta_+(\phi; s)$ and $\zeta_+^*(\phi; s)$ satisfy the functional equation of Hecke type.

(When $\boldsymbol{p}$ is even, we consider the second row.)

Holomorphic modular forms

Assume that $m-p$ is even. We define holomorphic functions $F(z)$ and $G(z)$ on $\mathcal{H}$ by

$$F(z) = (-1)^{\frac{m-p}{2}}(2\pi)^{-\frac{m}{2}} \cdot \Gamma\left(\frac{m}{2}\right) \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g$$
$$+ |D|^{-1/2} \cdot \sum_{n=1}^{\infty} M(P; n)\mathrm{e}[nz],$$
$$G(z) = i^{-\frac{m}{2}} \cdot (2\pi)^{-\frac{m}{2}} \cdot \Gamma\left(\frac{m}{2}\right) N^{\frac{m}{4}} |D|^{-1/2} \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g$$
$$+ (-1)^{\frac{m-2p}{4}} \cdot N^{\frac{m}{4}} \cdot \sum_{n=1}^{\infty} M^*(\widehat{P}; n)\mathrm{e}[nz].$$

Holomorphic modular forms

Theorem 12

Then, $F(z)$ and $G(z)$ are holomorphic modular forms for $\Gamma_0(N)$ of weight $m/2$. Further we have

$$F\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-m/2} = G(z).$$

This result is consistent with a result of Siegel in 1948, in which Siegel calculated the action of certain differential operators on indefinite theta series, and proved that in the case of $\det Y > 0$, we can construct holomorphic modular forms from indefinite theta series associated with Y.

Thank you very much!